Physics 200-04
Heisenberg Uncertainty Theorem

Having shown that the average value of the values determined for some observable A in state $|\psi\rangle$ is $\langle\psi|A|\psi\rangle$ we can define the so called uncertainty of the observable A is

$$\Delta A^2 = \langle\psi| \left(A - \langle\psi|A|\psi\rangle I\right)^2 |\psi\rangle \tag{1}$$

This is just the standard deviation of the probabilistic outcomes for the determination of A.

Let us call the matrix $\hat{A} = A - \langle\psi|A|\psi\rangle I$. Then $\Delta A^2 = \langle\psi| \left(\hat{A}\right)^2 |\psi\rangle$ and similarly for B. Now consider the matrix

$$C = (\hat{A} + i\lambda\hat{B}) \tag{2}$$

where λ is some real variable. Both A and B are assumed to be Hermitean matrices. C will not be Hermitean. However, for the un-normalized vector

$$|\phi\rangle = C|\psi\rangle \tag{3}$$

then we know that $\langle\phi||\phi\rangle = |\phi\rangle^\dagger|\phi\rangle$ is a real number greater than zero since it is just the sum of the absolute value squared of each element of the column vector $|\phi\rangle$. Thus

$$\begin{aligned} 0 &< (C|\psi\rangle)^\dagger C|\psi\rangle = \langle\psi|C^\dagger C|\psi\rangle \\ &= (\langle\psi|\hat{A}^2 + i\lambda(\langle\psi|i\hat{A}\hat{B}|\psi\rangle - \langle\psi|\hat{B}\hat{A}|\psi\rangle) + \lambda^2\langle\psi|\hat{B}^2|\psi\rangle \end{aligned} \tag{4}$$

. Since this is true for all real values of λ it must be true for the value of λ which minimizes this expression. That minimum occurs when

$$\lambda = -\frac{\langle\psi| \left(i(\hat{A}\hat{B} - \hat{B}\hat{A})\right) |\psi\rangle}{2\langle\psi|\hat{B}^2|\psi\rangle} \tag{5}$$

One worry is that the expression for λ looks as if it might be imaginary. But the matrix $i(\hat{A}\hat{B} - \hat{B}\hat{A})$ is Hermitean since

$$\begin{aligned} \left(i(\hat{A}\hat{B} - \hat{B}\hat{A})\right)^\dagger &= -i\left((\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger\right) \\ &= -i(\hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger) = i(-\hat{B}\hat{A} + \hat{A}\hat{B}) = i(\hat{A}\hat{B} - \hat{B}\hat{A}) \end{aligned} \tag{6}$$

Thus its eigenvalues are all real, its average value must be real, and λ is real.

Substituting this minimum value of lambda in, we get

$$
\begin{aligned}
0 &< \langle\psi|\hat{A}^2|\psi\rangle - \frac{\langle\psi|\left(i(\hat{A}\hat{B}-\hat{B}\hat{A})\right)|\psi\rangle}{2\langle\psi|\hat{B}^2|\psi\rangle}\langle\psi|\left(i(\hat{A}\hat{B}-\hat{B}\hat{A})\right)|\psi\rangle \\
&+ \left(\frac{\langle\psi|\left(i(\hat{A}\hat{B}-\hat{B}\hat{A})\right)|\psi\rangle}{2\langle\psi|\hat{B}^2|\psi\rangle}\right)^2\langle\psi|\hat{B}^2|\psi\rangle \\
&= \langle\psi|\hat{A}^2|\psi\rangle - \frac{\left(\langle\psi|\left(i(\hat{A}\hat{B}-\hat{B}\hat{A})\right)|\psi\rangle\right)^2}{4\langle\psi|\hat{B}^2|\psi\rangle} \qquad (7)
\end{aligned}
$$

Multiplying through by the positive number $\langle\psi|\hat{B}^2|\psi\rangle$ and rearranging, we have

$$
\langle\psi|\hat{A}^2|\psi\rangle\langle\psi|\hat{B}^2|\psi\rangle \geq \frac{1}{4}\left(\langle\psi|\left(i(\hat{A}\hat{B}-\hat{B}\hat{A})\right)|\psi\rangle\right)^2 \qquad (8)
$$

Note that

$$
\begin{aligned}
[\hat{A},\hat{B}] &= (A- <A> I)(B- <B> I) - (B- <B> I)(A- <A> I) \\
&= AB - A<B> - B<A> + <A><B> I \\
&\quad -(BA - B<A> - A<B> + <A><B> I) \\
&= AB - BA = [A,B] \qquad (9)
\end{aligned}
$$

Thus we have the Heisenberg uncertainty relation

$$
\Delta A^2 \Delta B^2 \geq \frac{1}{4}\langle\psi|i[A,B]|\psi\rangle^2 \qquad (10)
$$

Ie, the uncertainty (standard deviation) of A times that of B is always greater than the square of the expectation value of the commutator of A and B.

For our finite dimensional system, one consequence of this theorem is that the expectation value of the commutator in the eigenstate of of one of the operators of the commutator is always equal to zero. Ie, if the state $|\psi\rangle$ is an eigenstate of A , then ΔA is zero, and therefor $\langle\psi|i[A,B]|\psi\rangle = 0$. Ie, the commutator must always have both positive and negative eigenvalues (if it has non-zero eigenvalues).

For an infinite dimensional system (like for the postion matrix or momentum matrix for a free particle) the commutator can be positive (eg, proportional to the identity). In that case, this theorem shows that no eigenvectors for the operators in the commutator exist, and that the uncertainties of the operators in the commutator can never be zero.