"Twin's paradox for Ordinary distances."

Note that this analogy may help some people in grappling with the twin's paradox, and for others it will simply confuse them. If you find yourself in the latter boat, forget about this. It is certainly not required and is here purely as a help.

The Twin's paradox is usually phrased something like " If Bob, the person who moves away and then comes back, ages less than Alice, the person who stays at home, is claimed to age less by analyzing his clock from Alice's frame, why cannot we analyze Alice's clock from Bob's frame, and argue that it is Alice who should age less. After all by relativity, either Alice or Bob should be equally good observers. Thus, where did the asymmetry enter into this problem?"

Let us go through the same kind of analysis in terms of ordinary distances. In this case we are not doing special relativity. We are in three dimensional space and looking at how far Alice and Bob travel. Alice and Bob can always line up their coordinate axes with the direction in which they travel. In that case, the distance they travel is just equal to the difference in the x coordinate along their path. We will call this the "x-distance" of travel. Now consider Bob's coordinates to be the x, y frame, and Alice's the x', y'. Bob is located at $x = 0$. Now, Alice's coordinates are related to Bob's by a rotation

$$x' = cos(\theta)x + sin(\theta)y \tag{1}$$

$$y' = cos(\theta)x - sin(\theta)y \tag{2}$$

which can also be written as

$$x' = \Gamma(x + \kappa y) \tag{3}$$

$$y' = \Gamma(y - \kappa x) \tag{4}$$

where

$$\Gamma = \frac{1}{\sqrt{1 + \kappa^2}} \tag{5}$$

and $\kappa = \sin(\theta)/\cos(\theta)$ is the "slope" of the one system with respect to the other.

Now, let us look at the distance x' for a given lapse of distance x in Bob's frame. We find, since $y = 0$ in Bob's frame, that

$$x' = \frac{x}{\sqrt{1 + \kappa^2}} \tag{6}$$

or

$$x = \sqrt{1 + \kappa^2}x' \tag{7}$$

Thus, the x-distance measured by Alice is shorter than that measured by Bob, for this particular path. (This is called x-distance expansion.)

Now consider a path where Bob (according to Alice) travels out a for an x-distance of X with slope κ turns around and comes back again to Alice with a slope of $-\kappa$. On each journey, Alice will argue that Bob's x-distance measure is $\sqrt{1+\kappa^2}$ longer than Alice's. Thus on return Bob's x-distance will be $\sqrt{1+\kappa^2}$ longer than Alice's. Thus Bob on his return with his odometer, which measures his x-distance, reading a value longer than Alice's.

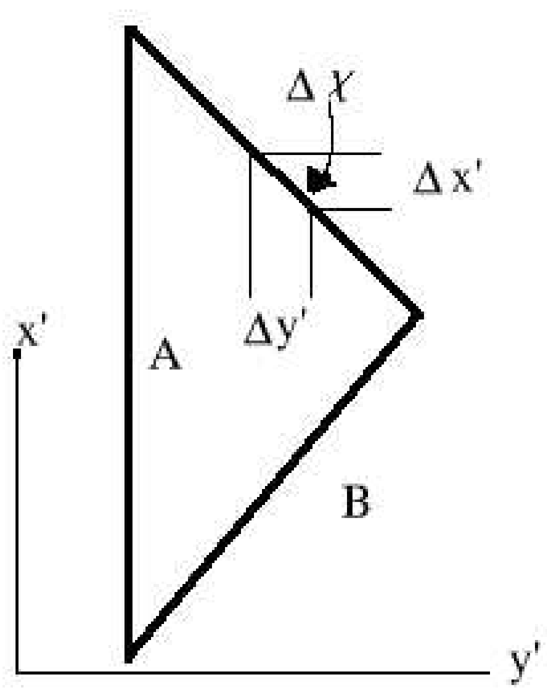

Figure 1: The plot of Bob and Alice's paths in Alice's x', y' system of coordinates.

Now however a troublemaker comes along and says– "Look by rotation invariance there is no difference between Alice and Bob. Thus Bob should regard Alice on each leg of the trip as having an x-distance of $\sqrt{1+\kappa^2}$ longer than Bob's just by relativity of rotation. Thus each should be able to argue that the other's journey was longer, which is a contradiction and shows that rotation relativity is non-sense."

Of course I point out that Bob suffered a change in direction of this path, while Alice did not. The troublemaker says-"How in the world can that tiny change in direction, which took up essentially none of the path that Bob traveled result in such a huge change in distance?"

Of course it was not that jag which in and of itself caused the change in distance. It was that jag as a symptom that Bob had followed a different path than Alice had which resulted in Bob's path being longer than Alice's.

We notice that we can write the change in x as a function of the change in x' as

$$\Delta x = \sqrt{1+\kappa^2}\Delta x' = \sqrt{\Delta x'^2 + (\kappa \Delta x')^2} = \sqrt{\Delta x'^2 + \Delta y'^2} \tag{8}$$

since $\frac{\Delta y'}{\Delta x'} = \kappa$. $\Delta x'$ and $\Delta y'$ are the components of a little piece of the path that some object follows in Alice's coordinate system. Thus we can call $\Delta\chi = \sqrt{\Delta x'^2 + \Delta y'^2}$ the "proper" distance that an object travels in two dimensional space. While we derived the above under assumption that the slope κ was a constant, we can generalize this concept of distance, call it the proper distance, which would be the sums of the $\Delta\chi$ everywhere along the path, even if the slope of the path $\frac{\Delta y'}{\Delta x'}$ is not everywhere constant. This χ distance is the x-distance that someone who aligned their x axis with the path at each point along the path would measure for that little piece of the path. This proper length of the path would be independent of which coordinate system we measured it in. It is an invariant under rotation.

The above is in fact exactly the way in which we define distances in a two dimensional space. χ, the "proper" distance is just what we call the distance, as opposed to the x-distance, along a curve. Furthermore this distance is not simply a function of the end points of the curve, but is a function of the whole curve. The jag, at which B changes direction is not the cause of the longer length of the B curve, it is simply one symptom of the fact that the B curve differs from the A curve. It is because it is different that it has a different length, not because of the existence of the change in direction of the curve.

Ie, All of the question one can ask about the proper time, about the twin's paradox, etc, are also questions one can ask about ordinary distances, and the twin's paradox of x-distances. The only difference is that the distance function for space-time has a strange Pythagoras' theorem since the hypotenuse is not the sum of the squares of the sides, but rather a difference in the squares of the sides. This means that, in case of the proper time, the distance through space-time, a straight, no-jagged path is longer, rather than shorter (as it is in ordinary space) than a jagged path. But the argument is almost identical in the two cases, except for that minor matter of a difference in signs in the Pythagoras' theorem.