Physics 200-04
Time and Velocity

Time:

As mentioned at the end of the Einstein notes, a clock in a fixed frame, when measured in a moving frame, runs at a rate given by

$$\delta\tau = \frac{\delta t}{\gamma} \tag{1}$$

where τ is the time in the frame moving with the clock (the frame in which the clock is at rest) and δt is time in the frame in which one is moving with a velocity v with respect to the clock.

This can be written as

$$\begin{aligned}
\Delta\tau &= \sqrt{1-\left(\frac{v}{c}\right)^2}\Delta t && (2)\\
&= \sqrt{\Delta t^2 - \frac{1}{c^2}(\vec{v}\Delta t)\cdot(\vec{v}\Delta t)} && (3)\\
&= \frac{1}{c}\sqrt{c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)} && (4)
\end{aligned}$$

Here $\Delta x, ..., \Delta t$ are just the changes in $x, ..., t$ produced by the motion of the clock.

But that factor in the square root looks very much like a distance, or rather a generalization of a distance, since instead of all plus signs as a distance would have by Pythagoras' theorem, it has some positive and some negative signs. However, if one ignores that point, one could argue that that expression **is** a distance, and that then what this expression expresses is that what the clock in the moving frame is measuring is a form of distance. As with any distance, this one is invariant under coordinate changes. In particular, we derived the Lorentz equations precisely under the assumption that this expression was kept invariant (the same) under the allowed changes in coordinates. Thus, accepting this as an expression for a kind or distance, a space-time distance, the Lorentz transformations are precisely those transformations which keep this distance the same.

Note also that we derived the above expression for the time as measured by the clock under the assumption that the clock was in some fixed frame moving with a constant velocity. However, now that we have recognized that the time on the moving clock is just the distance in the space-time, we can generalize this and assume that clocks, no matter what their state of motion– whether in uniform motion or not– always measure just this distance in space-time.

This distance through which a clock, or any object travels through the four dimensional space-time is called the proper time of the object. It represents by calculation in the uniform motion case, and by assumption in the general case, the time which would be measured by a clock dragged along with that object. It represents the time attached to the object.

Now the "twin's" paradox is completely sensible. Just as the distance from Vancouver to Toronto depends entirely on the path followed to get there (eg, via Dallas or via Winnipeg), and not just on the endpoints (Vancouver, Toronto). Similarly two clocks which follow two different paths through space-time will measure different distances. Clocks are like odometers– different trips imply different times, no matter that the endpoints of the trip might be the same.

Addition of velocities:

Since both lengths and times change from frame to frame, it is not surprising that velocities do as well, being just the ratio of distances over times. Thus consider a particle with constant velocity $\vec{w}$. The position of that particle will be $\vec{x} = \vec{x}_0 + \vec{w}t$. Plugging this into the Lorentz transformation for positions, we get

$$x' = \gamma(x_0 + w_x t - vt) \tag{5}$$

$$t' = \gamma(t - \frac{v}{c^2}(x_0 + w_x t)) \tag{6}$$

$$y' = y_0 + w_y t \tag{7}$$

$$z' = z_0 + w_z t \tag{8}$$

Using the second equation to express t in terms of t' so we can figure out the equation of motion of the particle in the primed system, we get

$$x' = x'_0 + \frac{(w_x - v)}{1 - \frac{vw_x}{c^2}} t' \tag{9}$$

$$y' = y'_0 + \frac{w_y}{\gamma\left(1 - \frac{vw_x}{c^2}\right)} t' \tag{10}$$

$$z' = z'_0 + \frac{w_z}{\gamma\left(1 - \frac{vw_x}{c^2}\right)} t' \tag{11}$$

where $x'_0,\ y'_0,\ z'_0$ are constants (independent of t') which depend on x_0, y_0, z_0 and v.

Thus the velocities in the primed frame are

$$w'_x = \frac{(w_x - v)}{1 - \frac{vw_x}{c^2}} \tag{12}$$

$$w'_y = \frac{w_y}{\gamma\left(1 - \frac{vw_x}{c^2}\right)} \tag{13}$$

$$w'_z = \frac{w_z}{\gamma\left(1 - \frac{vw_x}{c^2}\right)} \tag{14}$$

Ie, all three of the velocities in the x, y, z directions change from the one frame to the moving frame.

The transformation of velocities is rather a mess. We can in fact simplify it considerably, if instead of the ordinary velocity we define a new (so called proper) velocity in terms of the derivative with respect to proper time rather than with respect to derivative with respect to the time in the frame. This has the huge advantage that the proper time is an invariant. Ie, the proper time is the same no matter which coordinate system one looks at it.

The proper time for the particle is

$$d\tau = \sqrt{1 - \frac{w^2}{c^2}} dt \tag{15}$$

Thus

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}} \tag{16}$$

The proper velocity, which is the change in a position with respect to proper time, can therefor be written as

$$u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}} \tag{17}$$

$$u^x = \frac{dx}{d\tau} = w_x \frac{dt}{d\tau} = \gamma_w w_x \tag{18}$$

$$u^y = \frac{dy}{d\tau} = w_y \frac{dt}{d\tau} = \gamma_w w_y \tag{19}$$

$$u^z = \frac{dz}{d\tau} = w_z \frac{dt}{d\tau} = \gamma_w w_z \tag{20}$$

$$\tag{21}$$

where $\gamma_w = 1/\sqrt{1 - w^2/c^2}$. This definition of velocity is in fact far more convenient than is the usual definition, because τ does not change depending on which frame we are in. Therefor if we change frame, u^t, u^x, u^y, u^z change in exactly the same way as t, x, y, z change in a change of frame.

$$u'^t = \gamma_v (u^t - \frac{v u_x}{c^2}) \tag{22}$$

$$u'^x = \gamma_v (u^x - v) \tag{23}$$

$$u'^y = u^y \tag{24}$$

$$u'^z = u_z \tag{25}$$

and where $w^i = \frac{u^i}{u^t}$. The ordinary velocities w change in a very complicated way with a frame transformation, while the proper velocities change purely via a Lorentz transformation.

One interesting feature of the proper velocities is that they obey the equation

$$(u^t)^2 c^2 - \Big((u^x)^2 + (u^y)^2 + (u^z)^2\Big) = c^2 \tag{26}$$

And they obey this equation in all frames.

If we are interested in something moving at the velocity of light, then

$$c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0 \tag{27}$$

and we cannot define a proper time for this type of particle. In this case, we can take any parameterization of the path the particle follows to define the "proper" velocity. Ie, τ is undefined for this kind of particle, and we can choose it arbitrarily. Preferably we would choose it so that it is invariant under Lorentz transformations, but it does not really matter. In any case the length squared of a light-like proper velocity is zero.

Thus the proper, rather than the conventional, velocities have a number of very nice features. They obey the same transformation laws as the coordinates do, they obey a very simple constraint equation, and the velocities are easily derived from them.

We will find out in the future that it is the proper velocities which give us the best definition of Energy and momentum as well, rather than the velocities defined with respect to the frame time.

Aberration of light.

Consider the light coming to the earth from a distant star. If the earth were at rest (in the frame of the solar system), define the direction to the star by the angle θ that the light ray coming from the star makes with respect to direction of the (future)velocity vector of the earth, which we will take to be the x axis. Then the velocity of the light coming from the star to the earth will have components

$$w_x = -c\cos(\theta) \tag{28}$$
$$w_y = -c\sin(\theta) \tag{29}$$

Now assume that the earth is moving with velocity v along the x axis. In the new frame, the velocity of light is still c. In the new frame

$$w'_x = -c\cos(\theta') = \frac{-c\cos(\theta) - v}{\left(1 + \frac{c\cos(\theta)v}{c^2}\right)} \tag{30}$$
$$w'_y = -c\sin(\theta') = \frac{-c\sin(\theta)}{\gamma\left(1 + \frac{c\cos(\theta)v}{c^2}\right)} \tag{31}$$

If we look at the expression

$$\frac{\sin(\theta')}{1+\cos(\theta')} = \frac{1}{(1+\frac{v}{c})\gamma}\frac{\sin(\theta)}{1+\cos(\theta)} \tag{32}$$

But $\frac{1}{(1+\frac{v}{c})\gamma} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$ and

$$\frac{\sin(\theta')}{1+\cos(\theta')} = \frac{\sin(2(\theta'/2))}{1+\cos(2(\theta'/2))} \tag{33}$$
$$= \frac{2\sin(\theta'/2)\cos(\theta'/2)}{(2\cos^2(\theta'/2) - 1 + 1)} \tag{34}$$
$$= \tan(\theta'/2) \tag{35}$$

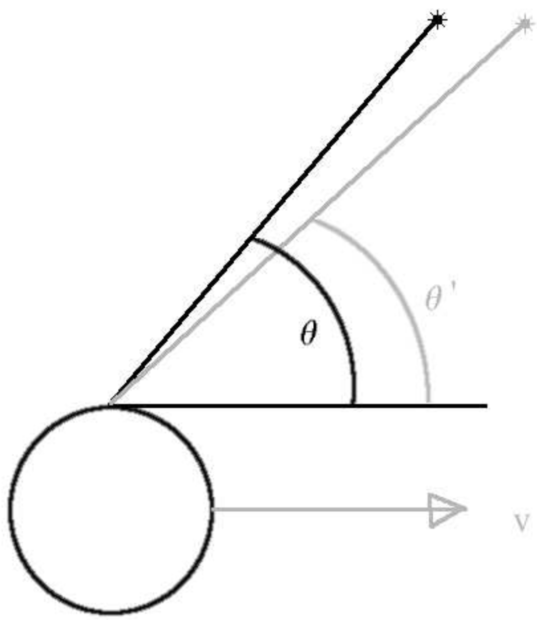

Figure 1: The angles defining the aberration of light. The solid angle is the direction to the star if the earth were at rest, while the light angle is that for the moving earth (let us say with respect to the center of mass of the solar system.)

Thus the equation for the aberration of light is

$$\tan(\theta'/2) = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tan(\theta/2) \tag{36}$$

For small angles looking in the direction in which the body is moving where $\tan(\theta/2) \approx \theta/2$, we have

$$\theta' \approx \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \theta \tag{37}$$

In the backward direction, near $\theta = \pi$, defining $\phi = \pi - \theta$ we have $\tan(\pi/2 -$

$\phi/2) = \cot(\phi/2) = 1/\tan(\phi/2)$, we have

$$\tan(\phi'/2) = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\tan(\phi/2) \tag{38}$$

Which is exactly what one would expect as one can regard the backward direction as the same as the forward direction but with v replaced with $-v$.

General boost

In the above we have consistantly just make a boost in the x direction. A boost in a general direction can be easily written. Assume that the direction of the velocity vector is $\vec{n}$– ie, $\vec{v} = v\vec{n}$.

$$t' = \gamma(t - \frac{\vec{v}\cdot\vec{x}}{c^2}) \tag{39}$$

$$\vec{n}\cdot\vec{x}' = \gamma(\vec{n}\cdot\vec{x} - vt) \tag{40}$$

$$\vec{n}\times\vec{x}' = \vec{n}\times\vec{x} \tag{41}$$

or

$$t = \gamma(t' + \frac{\vec{v}\cdot\vec{x}'}{c^2}) \tag{42}$$

$$\vec{n}\cdot\vec{x} = \gamma(\vec{n}\cdot\vec{x}' + vt) \tag{43}$$

$$\vec{n}\times\vec{x} = \vec{n}\times\vec{x}' \tag{44}$$