Physics 200-05

Practice problems 4– Two system quantum mechanics

In the following $\Sigma$ are the Pauli matrices for one particle and $\Xi$ are the Pauli matrices for the other particle. The same symbol $I$ will be used for the identity matrix of both.

1. Show that the matrix $\Sigma_2$ acting on the eigenvectors $|+1;3\rangle$ and $|-1;3\rangle$ of $\Sigma_3$ obey

$$\begin{aligned}\Sigma_2|+1;3\rangle = i|-1;3\rangle \\ \Sigma_2|-1;3\rangle = -i|+1;3\rangle\end{aligned} \tag{1}$$

This propety of the matrices acting so as to change one state vector into another is why they are often also called "Operators". Ie, the $\Sigma$ matrices are often called the $\Sigma$ opertors. In this context, the word Operator and the word Matrix are synonymous.

(In the infinite dimensional case, it is sometimes more convenient to call them operators than think of them as matrices. For example, if we look at the way in which the $P$ momentum matrix changes a state expressed in terms of the $X$ eigenvectors, $|\Psi\rangle = \int \psi(x)|x\rangle dx$, then

$$P|\Psi\rangle = -i\hbar \int \frac{\partial \psi(x)}{\partial x}|x\rangle dx \tag{2}$$

and the infinite dimensional $P$ matrix acts like the derivative operator on the coeficients of the state. )

Show that

$$\begin{aligned}\Sigma_2 \otimes \Xi_2(|1;3\rangle \otimes |-1;3\rangle) &= (\Sigma_2|1;3\rangle) \otimes (\Xi_2|-1;3\rangle) \\ &= |-1;3\rangle \otimes |+1;3\rangle\end{aligned} \tag{3}$$

and that

$$\begin{aligned}\Sigma_2 \otimes \Xi_2(|-1;3\rangle \otimes |+1;3\rangle) &= (\Sigma_2|-1;3\rangle) \otimes (\Xi_2|+1;3\rangle) \\ &= = |+1;3\rangle \otimes |-1;3\rangle\end{aligned} \tag{4}$$

Thus show that if

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1;3\rangle \otimes |-1;3\rangle - |-1;3\rangle \otimes |+1;3\rangle\right) \tag{5}$$

then

$$\Sigma_2 \otimes \Xi_2 |\psi\rangle = -|\psi\rangle \tag{6}$$

Ie, since we showed in class that $|\psi\rangle$ is an eigenstate of $\Sigma_1 \otimes \Xi_1$ and of $\Sigma_3 \otimes \Xi_3$, it is and eigenstate of all of the $\Sigma_i \otimes \Xi_i$. One can similarly show that if we define $\tilde{\Sigma} = \vec{\beta} \cdot \Sigma$ and $\tilde{\Xi} = \vec{\beta} \cdot \vec{\Xi}$ then $\tilde{\Sigma} \otimes \tilde{\Xi}|\psi\rangle = -|\psi\rangle$ This state is called the "singlet" state of two spin-1/2 particles.

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$$\Sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{7}$$

and

$$\begin{aligned} |1;3\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-1;3\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} \Sigma_2|1;3\rangle &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|-1;3\rangle \\ \Sigma_2|-1;3\rangle &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i|1;3\rangle \end{aligned} \tag{9}$$

Thus

$$\begin{aligned} \Sigma_2 \otimes \Xi_2(|1;3\rangle \otimes |-1;3\rangle) &= (\Sigma_2|1;3\rangle) \otimes (\Xi_2|-1;3\rangle) \\ &= (i|-1;3\rangle) \otimes (-i|1;3\rangle) = |-1;3\rangle \otimes |1;3\rangle \\ \Sigma_2 \otimes \Xi_2(|-1;3\rangle \otimes |1;3\rangle) &= (\Sigma_2|-1;3\rangle) \otimes (\Xi_2|1;3\rangle) \\ &= (-i|1;3\rangle) \otimes (i|-1;3\rangle) = |1;3\rangle \otimes |-1;3\rangle \end{aligned} \tag{10}$$

Thus

$$\begin{aligned} \Sigma_2 \otimes \Xi_2|\psi\rangle &= \frac{1}{\sqrt{2}}\left((\Sigma_2 \otimes \Xi_2)(|1;3\rangle \otimes |-1;3\rangle) - (\Sigma_2 \otimes \Xi_2)(|-1;3\rangle \otimes |1;3\rangle)\right) \\ &= \frac{1}{\sqrt{2}}\left(-|-1;3\rangle \otimes |1;3\rangle + |1;3\rangle \otimes |-1;3\rangle\right) = -|\psi\rangle \end{aligned} \tag{11}$$

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2)By the rules of multiplication of direct product matrices,

$$((\langle\psi|\otimes\langle\phi|)(|\tilde{\psi}\rangle\otimes|\tilde{\phi}\rangle) = (\langle\psi||\tilde{\psi}\rangle)\otimes(\langle\phi||\tilde{\phi}\rangle) = (\langle\psi||\tilde{\psi}\rangle)(\langle\phi||\tilde{\phi}\rangle) \tag{12}$$

Since the direct product of two numbers is just the ordinary product of the two numbers.

Show that if either $|\phi\rangle$ is orthogonal to $|\tilde{\phi}\rangle$ **or** $|\psi\rangle$ is orthogonal to $|\tilde{\psi}\rangle$, then the vectors $|\psi\rangle\otimes|\phi\rangle$ and $|\tilde{\psi}\rangle\otimes|\tilde{\phi}\rangle$ are orthogonal to each other.

$$\begin{aligned}((\langle\psi|\otimes\langle\phi|)(|\tilde{\psi}\rangle\otimes|\tilde{\phi}\rangle) &= (\langle\psi||\tilde{\psi}\rangle)(\langle\phi||\tilde{\phi}\rangle)\\ &= 0\end{aligned} \tag{13}$$

if either $\langle\psi||\tilde{\psi}\rangle = 0$ or $\langle\phi||\tilde{\phi}\rangle = 0$.But, $((\langle\psi|\otimes\langle\phi|)(|\tilde{\psi}\rangle\otimes|\tilde{\phi}\rangle) = 0$ is just the statement that $|\psi\rangle\otimes|\phi\rangle$ and $|\tilde{\psi}\rangle\otimes|\tilde{\phi}\rangle$ are orthogonal to each other. Ie, if either of the individual vectors in a product state are orthogonal then the product state is.

3)Show that

$$\langle\psi|\Sigma_3\otimes I|\psi\rangle = 0 \tag{14}$$

where $|\psi\rangle$ is as defined in question 1.

By extention all of the individual $\Sigma$ and $\Xi$ have zero expectation value. Thus, there is no state for the individual particles which captures the probabilities of this two particle entangled state.

Note that $\Sigma_3\otimes I$ is almost universally written as $\Sigma_3$, not because $\Sigma_3\otimes I$ equals $\Sigma_3$ (it does not and could not since $\Sigma_3\otimes I$ is a 4x4 matrix while $\Sigma_3$ is a 2x2) but because physicists do not like excess symbols. Ie, it is understood when talking about a two (or multi) particle system a single operator acts only on the states of that one particle and the other particle states are left alone.

$$\begin{aligned}&\langle\psi|\Sigma_3\otimes I|\psi\rangle\\ &= \frac{1}{2}(\langle 1;3|\otimes\langle -1;3| - \langle 1;3|\otimes\langle 1;3|)\left((\Sigma_3|1;3\rangle\otimes I|-1;3\rangle - (\Sigma_3|-1;3\rangle)\otimes I|1;3\rangle\right)\\ &= \frac{1}{2}\left((\langle 1;3|\Sigma_3|1;3\rangle)(\langle -1;3||-1;3\rangle) + (\langle -1;3|\Sigma_3|-1;3\rangle)(\langle 1;3||1;3\rangle)\right)\end{aligned} \tag{15}$$

where I have used the answer to problem 2 in getting to the last line. But $\langle 1;3|\Sigma_3|1;3\rangle = 1$ and$\langle -1;3|\Sigma_3|-1;3\rangle = -1$ so the two terms cancel.

4) Show that $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1;3\rangle \otimes |1,3\rangle + |-1;3\rangle \otimes |-1,3\rangle)$ is an eigenstate of $\Sigma_1 \otimes \Xi_1$ and $\Sigma_3 \otimes \Xi_3$ with eigenvalue $+1$, and thus would be a state for which the Bell's inequality expression would give a value of $+2\sqrt{2}$, which would also violate the classical inequality. Show that this state is also an eigenvector for $\Sigma_2 \otimes \Sigma_2$ with eigenvalue $-1$.

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$$\Sigma_3 \otimes \Xi_3 \Psi = \frac{1}{\sqrt{2}}\left((\Sigma_3|1;3\rangle) \otimes (\Xi_3|1;3\rangle) + (\Sigma_3|-1;3\rangle) \otimes (\Xi_3|-1;3\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left(((1)|1;3\rangle) \otimes ((1)|1;3\rangle) + ((-1)|-1;3\rangle) \otimes ((-1)|-1;3\rangle)\right) = |\Psi\rangle \quad (16)$$

Similarly using

$$\begin{aligned} \Sigma_1|1;3\rangle &= |-1;3\rangle \\ \Sigma_1|-1;3\rangle &= |1;3\rangle \end{aligned} \quad (17)$$

and similarly for $\Xi_1$, we have

$$\Sigma_1 \otimes \Xi_1 \Psi = \frac{1}{\sqrt{2}}\left((\Sigma_1|1;3\rangle) \otimes (\Xi_1|1;3\rangle) + (\Sigma_1|-1;3\rangle) \otimes (\Xi_1|-1;3\rangle)\, 4\right)$$

$$= \frac{1}{\sqrt{2}}\left((|-1;3\rangle) \otimes (|-1;3\rangle) + (|1;3\rangle) \otimes (|1;3\rangle)\right) = |\Psi\rangle \quad (18)$$

Thus this state is a $+1$ eigenvalue eigenstate of both the operators $\Sigma_3 \otimes \Xi_3$ and $\Sigma_1 \otimes \Xi_1$ .

But

$$\Sigma_2 \otimes \Xi_2 \Psi = \frac{1}{\sqrt{2}}\left((\Sigma_2|1;3\rangle) \otimes (\Xi_2|1;3\rangle) + (\Sigma_2|-1;3\rangle) \otimes (\Xi_2|-1;3\rangle)\, 4\right)$$

$$= \frac{1}{\sqrt{2}}\left(((i)|-1;3\rangle) \otimes ((i)|-1;3\rangle) + ((-i)|1;3\rangle) \otimes ((-i)|1;3\rangle)\right) = -|\Psi\rangle \quad (19)$$

Ie, it is an -1 eigenvalue eigenstate of $\Sigma_2 \otimes \Xi_2$

For Bell's thm, we took $A = \Sigma_1$, $B = \Sigma_3$, $C = \frac{1}{\sqrt{2}}(\Xi_1 + \Xi_3)$ and $D = \frac{1}{\sqrt{2}}(\Xi_1 - \Xi_3)$ Thus

$$\begin{aligned} &\langle\Psi|A \otimes C|\Psi\rangle + \langle\Psi|A \otimes D|\Psi\rangle + \langle\Psi|B \otimes C|\Psi\rangle - \langle\Psi|B \otimes D|\Psi\rangle \\ =\ & \langle\Psi|\left(A \otimes (C + D) + B \otimes (C - D)\right)|\Psi\rangle \\ =\ & \sqrt{2}\Psi\left(\Sigma_1 \otimes \Xi_1 + \Sigma_3 \otimes \Xi_3\right)|\Psi\rangle \\ =\ & \sqrt{2}\langle\Psi|\left(+1|\Psi\rangle + (+1)|\Psi\rangle\right) = 2\sqrt{2} \end{aligned} \quad (20)$$

which is greater than 2, the Bell classical limit at the positive end.

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