Physics 200-04

Pauli Spin Matrices

The two dimensional space (ie the space of physical qualities or attributes which can only have two possible values) is a particularly simply space in which everything can be solved exactly. Just as in classical mechanics the harmonic oscillator is the prime example of a physical system, which can both be solved easily and can be used as an approximation is a wide variety of situations, the two level system is the same for quantum mechanics. It can be solved exactly and can by used in a wide wide variety of physical situtions as a reasonable approximation.

One of the reasons is that the number of operators is very limited. A two by two matrix only has four complex entries, four complex numbers. If the matrix is furthermore Hermitean, then the two diagonal entries are real, and the off diagonal ones are complex conjugates of each other. Ie, a Hermitean two by two matrix only has four real numbers which characterise it.

Pauli defined a family of two by two Hermitean matrices in terms of which all others can be characterised. These are the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

, and three other matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

Any 2x2 Hermitean matrix can be written in terms of these three matrices

$$A = A_0 I + A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3 \tag{4}$$

where the A_i are real numbers. It is easy to see that this is a Hermitean matrix, and also that any Hermitean matrix can be written in this way.

For future consideration, let us define

$$\vec{A}\cdot\vec{\sigma} = A_1\sigma_1 + A_2\sigma_2 + A_3\sigma_3$$

Consider A and the two eigenvalues a_1 and a_2. Then it is straightforward to show that

$$a = A_0 \pm \sqrt{A_1^2 + A_2^2 + A_3^2} \tag{5}$$

Ie, the eigenvalues depend on on A_0 and the "length" of the three dimensional vector $\vec{A} = (\, A_1 \quad A_2 \quad A_3 \,)$. The eigenvectors are given by

$$\begin{aligned} |A, +\rangle &= \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \\ |A, -\rangle &= \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix} \end{aligned} \tag{6}$$

where the angles are defined by

$$\begin{aligned} A_1 &= |\vec{A}|\sin(\theta)\cos(\phi) = \sqrt{A_1^2 + A_2^2 + A_3^2}\sin(\theta)\cos(\phi) \\ A_2 &= |\vec{A}|\sin(\theta)\sin(\phi) && (7) \\ A_3 &= |\vec{A}|\cos(\theta) && (8) \end{aligned}$$

Ie, they are just the polar angles if we imagine A_1, A_2, A_3 to be the components of a spatial vector's xyz components. (Note that this works only if

$$\frac{A1^*}{A1} = \frac{A2^*}{A2} = \frac{A3^*}{A3} \tag{9}$$

—eg if A_i are either all real or all imaginary.)

Proof Using the above definition of the angles, we can write A as

$$\begin{aligned} A &= \begin{pmatrix} A_0 + |\vec{A}|\cos(\theta) & |\vec{A}|\sin(\theta)e^{-i\phi} \\ |\vec{A}|\sin(\theta)e^{i\phi} & A_0 - |\vec{A}|\cos(\theta) \end{pmatrix} \\ &= A_0 I + |\vec{A}| \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta) \end{pmatrix} \end{aligned} \tag{10}$$

where we recall that $\cos(\phi) + i\sin(\phi) = e^{i\phi}$.

Assuming that the eigenvector $|A, +\rangle$ is $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ the eigenvactor equation is matrix is

$$(A_0 + |\vec{A}|\cos(\theta))\alpha + |\vec{A}|\sin(\theta)e^{-i\phi}\beta = (A_0 + |\vec{A}|)\alpha$$

$$(A_0 - |\vec{A}|\cos(\theta))\beta + |\vec{A}|\sin(\theta)e^{i\phi}\alpha = (A_0 + |\vec{A}|)\beta \tag{11}$$

Solving the second for β

$$\beta = \frac{\sin(\theta)e^{i\phi}}{(1+\cos(\theta)}\alpha \tag{12}$$

recalling that $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$ and $1 + \cos(\theta) = 2\cos^2(\theta/2)$ we have

$$\beta = e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\alpha \tag{13}$$

which agrees with my former expression, if I take $\alpha = \cos(\theta/2)$. Note that this eigenvector has unit norm.

The solution for the other eigenvector follows just as easily

Consequences

The eigenvector depends on neither A_0 the multiple of the identity, nor $|\vec{A}|$ the length of the other parts of the matrix, but only on the "direction" $\frac{\vec{A}}{|\vec{A}|}$.

The expectation value $\langle A,+|\sigma_3|A,+\rangle$ of the larger eigenvalue of A with the matrix σ_3 is

$$\begin{aligned} &\begin{pmatrix} \cos(\theta/2) & e^{-i\phi}\sin(\theta/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \\ &= \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta) = \frac{A_3}{|\vec{A}|} \end{aligned} \tag{14}$$

With matrix σ_1 we get

$$\begin{aligned} &\begin{pmatrix} \cos(\theta/2) & e^{-i\phi}\sin(\theta/2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \\ &= \cos(\theta/2)\sin(\theta/2)(e^{i\phi} + e^{-i\phi}) = \sin(\theta)\cos(\phi) \\ &= \frac{A_1}{|\vec{A}|} \end{aligned} \tag{15}$$

and similarly for σ_2

$$\langle A,+|\sigma_2|A,+\rangle = \frac{A_2}{|\vec{A}|} \tag{16}$$

Thus, if $B = B_0 I + \vec{B} \cdot \vec{\sigma}$, we get

$$\langle A, +|B|A, +\rangle = B_0 + \vec{B} \cdot \frac{\vec{A}}{|\vec{A}|} \tag{17}$$

Similarly one can show that

$$\langle A, -|B|A, -\rangle = B_0 - \vec{B} \cdot \frac{\vec{A}}{|\vec{A}|} \tag{18}$$

Projection

We can define the matrix (in our case a 2x2 matrix) associated with the eigenvector, say $|A, +\rangle$ with

$$P_+ = |A, +\rangle\langle A, +| \tag{19}$$

Ie it is the product of the eigenvector by its dirac adjoint. This is a column matrix times a row matris, and produces a 2x2 matrix. Then

$$\begin{aligned} P_+P_+ &= |A, +\rangle\langle A, +||A, +\rangle\langle A, +| = |A, +\rangle(\langle A, +||A, +\rangle)\langle A, +| \\ &= (\langle A, +||A, +\rangle)|A, +\rangle\langle A, +| = |A, +\rangle\langle A, +| \\ &= P_+ \end{aligned} \tag{20}$$

since by assumption the eigenvectors are always chosen to the normalised. Any matrix whose square is itself is called a projection operator. Note that the matrix P_+ picks our the $|A, +\rangle$ eigenvalue part of a vector. If we have a general vector

$$|\psi\rangle = \alpha|A, +\rangle + \beta|A, -\rangle \tag{21}$$

then

$$P_+|\psi\rangle = \alpha|A, +\rangle\langle A, +||A, +\rangle + \beta|A, +\rangle\langle A, +||A, -\rangle = \alpha|A, +\rangle \tag{22}$$

Ie, it "projects out" the part of the vector $|\psi\rangle$ which is along the $|A, +\rangle$ direction.

We similarly also have the projection operator in the $P_- = |A, -\rangle\langle A, -|$ which is the projection operator onto the negative eigenstate.

The matrix A can be written as

$$A = a_+P_+ + a_-P_- \tag{23}$$

ie as teh sum of the projection operators associated with the various eigenvalues times the eigenvalue. This expression clearly has the same eigenvalues and eigenvectors that A has.

$$
\begin{aligned}
(a_+P_+ + a_-P_-)|A,+\rangle = a_+|A,+\rangle \\
(a_+P_+ + a_-P_-)|A,-\rangle = a_+|A,-\rangle
\end{aligned}
\tag{24}
$$

Thus

$$
\begin{aligned}
(a_+P_+ + a_-P_-)(\alpha|A,+\rangle + \beta|A,-\rangle) &= a_+\alpha|A,+\rangle + a_-\beta|A,-\rangle \\
&= A(\alpha|A,+\rangle + \beta|A,-\rangle)
\end{aligned}
\tag{25}
$$

and the multiplication of any arbitrary vector by the two matrices gives identical vectors. Ie, the difference between the two matrices must be zero.

Multiplication

The product of the σ matrices are simple

$$
\begin{aligned}
\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I \\
\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3 \\
\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1 \\
\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2
\end{aligned}
\tag{26}
$$

This implies that

$$
(\vec{n}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma}) = \vec{n}\cdot\vec{m} + i\vec{n}\times\vec{m}\cdot\vec{\sigma}
\tag{27}
$$