Physics 200-04
Momentum and Energy

**Forces and Newton's Laws**

Having cleared up the problem with light and aether, Einstein and the physics community were left with a rather large problem, namely Newton's laws. While Newton's first law was not problematic– a particle which traveled with constant velocity in one frame also traveled with constant (but different) velocity in another frame. Ie, if the particle traveled in a straight line in the absence of forces in one frame, it also did in another frame. However both the second and third laws were now in trouble. the second law stated that the internal kinematics of the particle, the acceleration in particular, was determined by something outside the particle, the forces exerted on the particle by external influences. One had two problems here.

The more benign was the acceleration. While the acceleration could have been defined in the same way as in ordinary Newtonian physics as the rate of change of the velocity with respect to time, it was immediately clear that this would have resulted in a mess. Since time, and simultenaity change from frame to frame, the transformation of acceleration defined in this way from frame to frame would have been an even greater a mess than the transformation of the velocity. Now time occurs twice in the denominator $a = \Delta(\Delta x)/(Deltat)^2$ and the transformation law of the time would make the denominator of the transformed acceleration in terms of the original and even greater mess than in the case of velocities. However, the definition of proper velocities gives on the key clue as to how to define the proper acceleration, namely with respect to the proper time of the particle, rather than the ordinary time of any observer. Since the proper time is an invariant– is the same in all reference frames, the denominator of the definition of proper acceleration does not change when reference frames are changed. It is only the numerator which changes, and it will change in exactly the same way as do the coordinates– since it is just the difference of coordinates. Ie, if the path of a particle is given by

$$t(\tau), x(\tau), y(\tau), z(\tau) \tag{1}$$

(since again the time in general will have a non-trivial dependence on the proper time), then the components of the proper acceleration will be

$$a^t = \frac{d^2t(\tau)}{d\tau^2} \tag{2}$$

$$a^x = \frac{d^2x(\tau)}{d\tau^2} \tag{3}$$

$$a^y = \frac{d^2y(\tau)}{d\tau^2} \tag{4}$$

$$a^z = \frac{d^2z(\tau)}{d\tau^2} \tag{5}$$

$$\tag{6}$$

Because $\tau$ is the path-length of the particle, and thus is not and entirely independent variable, just as the components of the proper velocity are not independent but obey

$$c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 = c^2 \tag{7}$$

so the acceleration also obeys a constraint, most simply derived by taking the derivative of the velocity constraint with respect to $\tau$, namely

$$c^2a^tu^t - a^xu^x - a^yu^y - a^zu^z = 0. \tag{8}$$

Note that in addition to spatial accelerations, there is also a temporal acceleration, $a^t$. In the frame in which the particle is at rest (only has a non-zero component of $u^t$), the temporal component of the acceleration is zero.

The second problem is the force. The above constraint on the acceleration also implies a constraint on the force (it must have a temporal component as well as the usual spatial components, and must satisfy the same constraint as the accelerations). But how does one figure out what the force is.

One of Newton's great breaks with prior tradition was in his handling of forces. Until that time forces were believed to be always be contact forces. One body exerted a force on another because the two bodies were in contact. Newton, with his law of gravity, postulated the existence of forces which acted at a distance. The force between two particles due to gravity was something which operated even if there was nothing between the bodies. It was proportional to the inverse distance squared between the bodies.

Descartes had postulated that gravity was transmitted by some kind of fluid, the planets being the locations of some sort of vortices within the fluid and swept on by the rotation of the fluid caused by the sun. Newton eliminated that, and replaced it with a direct influence of one body on the other.

The problem in special relativity is that this option is closed to one. If two bodies are supposed to influence each other across a distance, what aspect is it about the second body which causes the force? It cannot be the distance since the distance is a function of time, and one has lost the notion of "now". One cannot say, it is the distance apart now which causes the force, because now depends on which observer. Besides, if it were "now" then a person near the second body could transmit a signal to the first "instantly" by wiggling the second body, and the person near the first body need just look at the motion of the first body to see the signal. Ie, such a force would be acausal, and could thus be used to violate causality and signal into the past.

One could argue that it was the position of the second body at times long enough in the past that light could have traveled from the second body to the first (ie locations of the second body in the past light-cone of the first) which exert the force on the first body. But this makes it very difficult to see how the third law, and the conservation of energy and momentum could be true.

Fortunately Faraday had invented the concept of fields and Maxwell had used it in his theory of electromagnetism. The field is located everywhere and at all times. Local disturbances in the field affect only nearby values of the field. Particles can cause disturbances in the field, but only locally. Those disturbances can travel out from the first particle and affect other particles, due to the local interaction between the field and the particle. This field becomes very similar to the aether, or to Descarte's fluid.

**Momentum**

Instead of following this line of thought, Einstein went down a different road. One of the consequences of Newtonian physics was the concept of energy and of momentum, quantities which are supposed to be conserved during interactions between particles or between particles and fields. (Physicists from Maxwell on had shown that one could associate both momentum and energy with the Electromagnetic field). He then asked whether these concepts could be generalized to Special Relativity.

In particular one wants to keep the concept of conservation of momentum.

In ordinary Newtonian physics, one defines the momentum as $m\vec{v}$ and shows via Newton's laws that in any collision

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_3\vec{v}_3 + m_4\vec{v}_4 \tag{9}$$

where $m_1, m_2$ are the two masses before the collision and $m_3, m_4$ after. (Usually one would choose $m_3 = m_1$ and $m_4 = m_2$). It was assumed that mass, representing something inherent in the world which did not change, was also conserved in any such interaction $m_1 + m_2 = m_3 + m_4$.

The question Einstein asked was how one could generalize this to something which would also be conserved in any frame in Special Relativity. It is immediately clear that simply defining the momentum by $p = mv$ would not work very well. The transformation of the velocities into the moving frame is a mess, and the conservation of momentum in one frame would not necessarily imply the conservation in another frame. It is possible to do it, if one also allows the mass to change from frame to frame, but who wants a physics that is such a mess.

Instead, we can use the proper velocity $\vec{u} = \frac{d\vec{x}}{d\tau}$ instead to define the momentum. Ie, we define

$$\vec{p} = m\vec{u} \tag{10}$$

For small velocities the proper velocity is almost the same as the ordinary velocity, so this momentum would be almost the same as $m\vec{v}$. Furthermore, this velocity has a much simpler transformation law from one frame to the next than does the ordinary velocity, making the chance of preserving the conservation law much higher on transferring from one frame to the next. In fact, if $v_r$ is the velocity of the one frame with respect to the other (in the x direction)– it is not the velocity of the particle, it is the velocity of the frame– we have

$$u'^x = \gamma(u^x - v_r u^t) \tag{11}$$

$$u'^y = u^y \tag{12}$$

$$u'^z = u^z \tag{13}$$

The only problem is the in the transformed frame $u'^x$ depends not only on $u^x$ but also on $u^t$. The putative conservation law in the new frame, traveling with velocity $v_r$ in the x direction would be,

$$p'^x_1 + p'^x_2 = p'^x_3 + p'^x_4 \tag{14}$$

becomes

$$m_1\gamma(u_1^x - v_r u_1^t) + m_2\gamma(u_2^x - v_r u_2^t) = m_3\gamma(u_3^x - v_r u_3^t) + m_4\gamma(u_4^x - v_r u_4^t) \quad (15)$$

in terms of the old frame. By assumption the x-momenta are conserved in the old frame. In order for the momentum in the new frame to be conserved, not only must the momentum in the old frame be conserved, but we also need what is apparently a new law of physics, namely

$$m_1 u_1^t + m_2 u_2^t = m_3 u_3^t + m_4 u_4^t \quad (16)$$

But is this a new law of physics?

Recall that $\vec{v} = \frac{\vec{u}}{u^t}$ and so the constraint on the proper velocity,

$$c^2 u^{t2} - u^{x2} - u^{y2} - u^{z2} = c^2 \quad (17)$$

becomes

$$(u^t)^2(c^2 - \vec{v}\cdot\vec{v}) = c^2 \quad (18)$$

or

$$(u^t) = \frac{1}{\sqrt{1 - \frac{\vec{v}\cdot\vec{v}}{c^2}}} \quad (19)$$

Expanding in a Taylor series to lowest order in $v^2 = \vec{v}\cdot\vec{v}$

$$u^t \approx 1 + \frac{1}{2}\frac{\vec{v}\cdot\vec{v}}{c^2} = 1 + \frac{1}{2}\frac{v^2}{c^2} \quad (20)$$

Thus the "new" conservation law becomes

$$m_1(1 + \frac{1}{2}\frac{v_1^2}{c^2}) + m_2(1 + \frac{1}{2}\frac{v_2^2}{c^2}) \approx m_3(1 + \frac{1}{2}\frac{v_3^2}{c^2}) + m_4(1 + \frac{1}{2}\frac{v_4^2}{c^2}) \quad (21)$$

In Newtonian physics, we have two other conservation laws, besides momentum conservation, namely the conservation of mass and the conservation of kinetic energy in an elastic collision. The above "new" law is that a statement that what is required in special relativity, in order that we can define a conservation law for momentum, is that the sum of the mass plus the kinetic energy over $c^2$ be conserved.

This immediately leads to the suspicion that perhaps we do not need two conservation laws. Maybe there is only one, the conservation of time-like momentum $p^t = mu^t$. This also then suggests that perhaps mass is not conserved. Perhaps mass can be converted into kinetic energy as long as the total of mass plus kinetic energy over $c^2$ remains conserved. Ie, as long as ($mc^2$+K.E.) is conserved. From this point of view, $mc^2$ would just be another kind of energy, which perhaps could, under certain circumstances, be converted into Kinetic energy. Ie, a particle naturally not only has a kinetic energy, but also a kind of energy with is present even when the particle is at rest, a rest-energy, just equal to the mass times $c^2$.

If this is true, then any particle has associated with it a huge energy. Consider one gramme worth of matter. Since $c^2 = 10^{17} m^2/s^2$ the energy would be $10^14J$. One gramme of TNT releases about 8 Kcalories when it explodes (about the same as the energy in one gramme of fat in your diet– remember that one food calorie is one Kcalorie), which is about 30KJ. Thus the energy contained in one gramme of matter is about the same as that in 3 $10^9$g = 3Ktonne of TNT or of fat. Fortunately this energy does not convert to kinetic energy very easily.

Let us analyze a couple of situations. In the first, let us assume that we have a mass $M$ which splits into two equal pieces of mass $m$ which fly off with momentum $\vec{p}_1$ and $\vec{p}_2$. Since the particle at first is assumed to be at rest, the initial total momentum must be zero, and thus $\vec{p}_1 + \vec{p}_2 = 0$. The conservation of time-momentum is$p^1_1 + p^t_2 = M$ since $u^t_1 = 1$.

Now from the constraint on the velocity we have the constraint on the momentum for any particle of mass $m$

$$c^2(p^t)^2 - (p^x)^2 - (p^y)^2 - (p^z)^2 = m^2c^2 \tag{22}$$

Since the momenta of each of the final particles is the same, and their masses are the same, we have that $p^t$ must also be the same for each of the particles. Thus we have

$$M = 2p^t \tag{23}$$

The kinetic energy is the total time momentum times $c^2$ minus the rest-energy, thus giving

$$\text{TotalKE} = 2p^tc^2 - 2mc^2 = (M - 2m)c^2 \tag{24}$$

Thus if we can somehow persuade a large mass to split into two equal sized smaller masses, the difference in the masses will be converted to kinetic energy. This is what happens in the fission bomb. One uranium nucleus splits into two approximately equal sized nuclei at around a Beryllium or Krypton masses. But two of those nuclei have a smaller mass than a Uranium nucleus. That extra mass is converted into kinetic energy. The first atomic bomb, with an energy of about 20Ktonne of TNT thus converted about 10 grammes of mass of uranium into Kinetic energy.

**Light**.

The above has assumed that we are talking about particles with a time-like path, and a proper time. What happens if we are talking about a particle of light? It follows a null trajectory, a trajectory with zero length. Thus the proper time is zero and we cannot define a proper velocity, or rather the proper velocity would be infinity (change of distance divided by zero change in proper time). If we multiply this by zero, we can imagine getting a finite answer. Ie, we could imagine that the particle would have zero mass, but could still have finite energy and momentum. The constraint would give

$$c^2(p^t)^2 - \vec{p}\cdot\vec{p} = 0 \tag{25}$$

Or, if we define $p_t = \frac{E}{c^2}$ where $E$ is the total energy including the rest-energy, then $E = |p|c$. Fortunately theorists looking at Maxwell's equations had concluded that the electromagnetic field must have energy and must carry momentum. If one were to consider a wave packet– a non-zero disturbance in the field– traveling cohesively in one direction, they had already showed that the energy and momentum of that disturbance MUST have exactly that relation. Ie, it would be consistent to think of the disturbance as though it were a particle traveling at the speed of light, and carrying energy and momentum, and having zero mass.

Note that in all of these cases, we can define the velocity by

$$\vec{v} = \frac{\vec{p}}{p^t} = c^2\frac{\vec{p}}{E} \tag{26}$$

In that case of a massive particle this follows directly from the definition of the momentum in terms of the proper velocity, and in the case of light, it produces a velocity which has magnitude of $c$ and traveling in the direction of the momentum.

Let us look at another problem. a mass can emit light. An atom can decay emitting light, or a light-bulb can cool down by emitting light. Let us consider a mass $M$ emitting light in the x direction, and leaving behind a mass $m$. We will find that $m$ must be smaller than M, so I am not assuming that they are the same mass. Before the emission, the mass $M$ is assumed to be at rest, so the momentum is zero. As before the momentum of the light must therefor be equal and opposite the momentum of the mass $m$.

$$p_m^x = -p_L^x \equiv p \tag{27}$$

We also need

$$p_m^t + p_L^t = p_M^t = M \tag{28}$$

We also have that

$$c^2(p_m^t)^2 - p^2 = m^2c^2 \tag{29}$$
$$c^2(p_L^t)^2 - p^2 = 0 \tag{30}$$

Thus we have

$$c^2(M - |p|/c)^2 - p^2 = m^2c^2 \tag{31}$$

or

$$|p| = \frac{M^2 - m^2}{2M}c \tag{32}$$

Thus the energy carried off by the light is

$$E_L = |p|c = \frac{M^2 - m^2}{2M}c^2 \tag{33}$$

and the kinetic energy in the recoil of the remaining particle is

$$E_m - mc^2 = p_m^t c^2 - mc^2 = Mc^2 - E_L - mc^2 = \frac{(M-m)^2}{2M}c^2 \tag{34}$$

Ie, only a small fraction $(M-m)/2M$ of the difference in rest energy of the initial and final particle ($(M-m)c^2$) goes into the kinetic energy of the final particle. Almost all of the difference in energy goes into the emitted light.

**4-vectors**

In the above, I constantly differentiated between the time component of a quantity (eg momentum, proper velocity, proper acceleration, etc) and the spatial components. In the days of Maxwell, until the the early 20th century all equations were written differentiating between the various x,y,z components of ordinary vectors. It became apparent however that equations would become far simpler and compact if could define an abstract thing called a vector, for example $\vec{v}$. Similarly for four dimensional quantities, it is far simpler to write things in terms of a four dimensional vector, let us designate it by $\bar{u}$ for example. One can think of $\bar{u}$ as being an abstract notation for a 4-vector, or one can think of it for example in terms of matrices.

$$\bar{u} = \begin{pmatrix} u^t \\ u^x \\ u^y \\ u^z \end{pmatrix} \tag{35}$$

The elements of the (4rows, 1column) matrix are then the components of the vector.

Note that it is important not to confuse a symbol like $\bar{u}$ with $\vec{u}$. The former is a 4-vector with four (t,x,y,z) components. The latter is a 3-vector with 3 (x,y,z) components. And the dot products between the two is very different.

This 4-vector is useful, because the transformation of these 4-vectors can again be written in terms of matrices. A Lorentz transformation is given by

$$\bar{u}' = L\bar{u} \tag{36}$$

For example with the usual transformation with velocity $v$ in the $x$ direction, we would have

$$L = \begin{pmatrix} \gamma & -\gamma\frac{v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 * 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

and the above equation would be

$$\begin{pmatrix} u'^t \\ u'^x \\ u'^y \\ u'^z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1*0 & \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^t \\ u^x \\ u^y \\ u^z \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} \gamma(u^t - \frac{v}{c^2}u^x) \\ \gamma(-vu^t + u^x) \\ u^y \\ u^z \end{pmatrix} \tag{39}$$

In analogy with the usual inner product–dot product– of ordinary three vectors, we can define a dot product also for these 4-vectors where however we use the Pythagorian theorem for the four dimensional space

$$\bar{w} \cdot \bar{u} = c^2 w^t u^t - w^x u^x - w^y u^y - w^z u^z = -\bar{w}^T G \bar{u} \tag{40}$$

where the last expression is thinking of the 4-vectors as matrices and $G$ is the matrix we defined earlier in the course

$$G = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

(Note the switch in sign. This was because earlier we wanted to emphasize the similarity between the distance defined by relativity and ordinary distances while here we want to primarily work with timelike vectors, and as a result want them to have positive length squared).

We can thus see immediately that this dot product is preserved by Lorentz transformations.

$$\bar{w}' \cdot \bar{u}' = -(L\bar{w})^T G(L\bar{u}) = -\bar{w}^T (L^T G L)\bar{u} = -\bar{w} G \bar{u} = \bar{w} \cdot \bar{u} \tag{42}$$

Ie, this dot product between vectors is exactly the same in all frames. This can be very useful.

We define the length squared of any vector by the dot product

$$\text{length}^2(\bar{u}) = \bar{u} \cdot \bar{u} = c^2 u^{t2} - u^{x2} - u^{y2} - u^{z2} \tag{43}$$

Note that the length squared can be positive, negative, or zero depending on whether the vector is timelike, spacelike or null (lightlike).

There is no analogy of the cross product for these vectors. The cross product, an antisymmetric multiplication of two vectors which gives another vector, is only defined in three dimensions.

The conservation of 4-momentum can thus be written in terms of these 4-vectors as

$$\bar{p}_1 + \bar{p}_2 = \bar{p}_3 + \bar{p}_4 \tag{44}$$

Similarly the condition on $\bar{p}$ can be written as

$$\bar{p} \cdot \bar{p} = m^2 c^2 \tag{45}$$

This notation is highly compact, which, if you are carefull, can allow writing equations very compactly. Of course as with any compact notation, you must be careful to remember what it means.

**Decay revisited**

The use of such invariance allows us to solve the problem of the emission of light by a particle of mass M in a much more efficient manner. We have

$$\bar{p}_M = \bar{p}_m + \bar{p}_L. \tag{46}$$

We can rewrite this as

$$\bar{p}_M - \bar{p}_L = \bar{p}_m \tag{47}$$

Taking the dot product of each side with itself, we get

$$(\bar{p}_M - \bar{p}_L) \cdot (\bar{p}_M - \bar{p}_L) = \bar{p}_m \cdot \bar{p}_m \tag{48}$$

$$M^2 c^2 - 2\bar{p}_M \cdot \bar{p}_L + 0 = m^2 c^2 \tag{49}$$

Since the only component of $p_M$ is the time-like component, and it is equal to $M$, we have

$$M^2 c^2 - 2c^2 M p_L^t = m^2 c^2 \tag{50}$$

$$\text{or} \tag{51}$$

$$p_L^t = \frac{M^2 - m^2}{2M} \tag{52}$$

as before. Similarly using $\bar{p}_M - \bar{p}_m = \bar{p}_t$ and squaring both sides, we get

$$p_m^t = \frac{M^2 + m^2}{2M} \tag{53}$$

Recalling that $p^t = E/c^2$ in each case we obtain the same result but with significantly less work.

**Compton Scattering**

A situation which had immense importance to the field of physics was what is now called Compton Scattering. In another of the 1905 papers, now called the photoelectric effect paper, Einstein extended Planck's hypothesis that light was absorbed by matter in lumps of energy. Einstein hypothesised, against all evidence of over 100 years of physics experiment and theory, that this meant that light behaved as particles with exactly the lumping of energy $E = h\nu$ where $h$ is Planck's constant and $\nu$ is the frequency of the light. He predicted that this would imply a certain linear relation between the energy of the most energetic emitted electrons emitted from the surface of a metal when light shone on it and the frequency of the light, a linear relation whose slope should be exactly Planck's constant. While in the mid teens, Millikan had confirmed this relation and in 1919 Einstein was given the Nobel prize for this formula (but not for the theory), most physicists still did not believe that light was composed of particles. (Einstein thus got the Nobel prize for a theory no-one believed instead of for a theory, Special Relativity, which everyone but one member of the Nobel Physics committee believed.)

It was the experiments performed by Compton in the early 1920's that turned around most physicists. Compton shone high energy gamma rays, believed to ultra high frequency electromagnetic radiation emitted by nuclei, onto matter. He found that there was a definite relation between the angle at which the resulting gamma ray came out, and the angle and energy of the electron which was emitted at the same time. The overwhelming hypothesis was that this was due to the collision of a particle corresponding to the gamma ray and an electron in the material.

Thus, we can assume that we have a gamma ray particle with energy $\epsilon$ impinging on an electron at rest. That gamma ray is scattered at an angle $\theta$ to its original motion, and with a resultant energy $\epsilon'$. The electron, originally at rest, comes out with energy $E$ and at an angle $\phi$ to the original direction of the gamma ray.

While we can write down the eight equations for this system ( The conservation of energy, the three equations for the conservation of momentum, and the four equations for the rest masses of the intital and final photon and the intitial and final electron), and solve them by brute force, there is a far quicker method using 4-vectors.

Let $\bar{P}$ be the intial 4-momentum of the electron. Since it is at rest it will only have a $P_t$ component. The intial gamma ray, with energy $\epsilon$ will have a momentum. Assuming that we are using coordinates where $c = 1$, and assuming that the initial gamma ray is travelling along the positive x-axis, it will only have an x component of momentum $p_x = \epsilon$, since $\epsilon^2 - p_x^2 = 0$. (A gamma ray has zero rest mass). Writing the conservation equations

$$\bar{p} + \bar{P} = \bar{p}' + \bar{P}' \tag{54}$$

where $\bar{p}$ and $\bar{p}'$ are the 4-momentum of the initial and final gamma ray, and $\bar{P}$ and $\bar{P}'$ are the initial and final momentum of the electron.

The problem is that while the magnitude of the 3-momentum of the final gamma ray is known to equal the final energy $\epsilon'$, and by assumption the angle $\theta$ that this final 3-momentum makes with the x-axis is known, nothing is known of the final energy or momentum of the electron. We would therefor like to get rid of all of the unknowns having to do with this final electron. We can do so if we isolate it and take its 4-length, since we do know that the $\bar{P}' \cdot \bar{P}' = M^2$.

Thus we can write

$$\bar{P}' = \bar{P} + \bar{p} - \bar{p}' \tag{55}$$

Squaring both sides of this equation (ie taking the dot product with itself) we get

$$M^2 = \bar{P}' \cdot \bar{P}' = (\bar{P} + \bar{p} - \bar{p}') \cdot (\bar{P} + \bar{p} - \bar{p}') \tag{56}$$

$$= \bar{P} \cdot \bar{P} + \bar{p} \cdot \bar{p} + \bar{p}' \cdot \bar{p}' + 2\bar{P} \cdot \bar{p} - 2\bar{P} \cdot \bar{p}' - 2\bar{p} \cdot \bar{p}' \tag{57}$$

Now, on that right hand side we have

$$\bar{P} \cdot \bar{P} = M^2 \tag{58}$$

$$\bar{p} \cdot \bar{p} = \bar{p}' \cdot \bar{p}' = 0 \tag{59}$$

$$\tag{60}$$

since the initial electron has rest mass M and the gamma rays both have rest mass 0.

$$\begin{aligned}\bar{P}\cdot\bar{p} &= M\epsilon &(61)\\ \bar{P}\cdot\bar{p}' &= M\epsilon' &(62)\\ \bar{p}\cdot\bar{p}' &= \epsilon\epsilon' - \vec{p}\cdot\vec{p}' = \epsilon\epsilon' - |\vec{p}||\vec{p}'|cos(\theta) &(63)\\ &= \epsilon\epsilon' - \epsilon\epsilon' cos(\theta) &(64)\end{aligned}$$

Thus we have

$$M^2 = M^2 + 2M\epsilon - 2M\epsilon' - 2\epsilon\epsilon'(1-\cos(\theta)) \tag{65}$$

or

$$\epsilon' = \frac{M\epsilon}{M+\epsilon(1-cos(\theta))} \tag{66}$$

Ie, unless $\theta = 0$, $\epsilon'$ will always be less than $\epsilon$, or the energy of the gamma ray will be less after the scattering by an amount that depends on the angle through which it has scattered.

From the conservation of energy, we have that

$$E' = M + \epsilon - \epsilon' \tag{67}$$

which since $\epsilon' < \epsilon$ is always greater than $M-$ ie the electron picks up energy in the collision.

We can also find this by rewriting the 4-momentum equations as

$$\bar{P} - \bar{P}' = \bar{p}' - \bar{p} \tag{68}$$

and squaring both sides, to get

$$2M^2 - 2ME' = -2\bar{p}'\cdot\bar{p} = -2\epsilon\epsilon'(1-cos(\theta)) \tag{69}$$

or

$$E' = \frac{M^2 + \epsilon\epsilon'(1-cos(\theta))}{M} \tag{70}$$

Similarly we can find the angle $\phi$ by squaring both sides of

$$\bar{P}' - \bar{p} = \bar{P} - \bar{p}' \tag{71}$$

to give

$$M^2 - 2\bar{P}' \cdot \bar{p} = M^2 - 2M\epsilon \tag{72}$$

Using

$$\bar{P}' \cdot \bar{p} = \epsilon(E' - |\vec{P}'|cos(\phi)) \tag{73}$$

we get

$$\cos(\phi) = \frac{E' - M\epsilon}{|\vec{P}'|} \tag{74}$$

which we can calculate knowing $E'$ and $|\vec{P}'|^2 = E'^2 - M^2$.

Ie, using the various dot products of the 4-momenum, we can calculate all of the quantities of interest without having to solve the equations by brute force.

Compton's experimental results verified these equations, thus acting both as a verification of Einstein's 1905 special relativity paper and his light quantum hypothesis (since it was almost impossible to think of how these equations would be valid in the wave nature of light).