Physics 200-04

Dynamics

Having talked about the way in which attributes of a physica system are described within matrix mechanics, it is time to discuss the dynamics– how do things change. There turn out to be two, equivalent, ways of describing the dynamics. I will again start with Heisenberg's aproach, and go on to Schroedinger's approach later.

Heisenberg developed his quantum mechanics, matrix mechanics, by looking at the dynamics. He was trying to figure out how the electromagnetic radiation was produced by the atoms when they decayed. He used two pieces of information. When the atom jumps from one energy level, with energy E_1 to another E_2, it emits light. By Einstein's hypothesis, that light, which must have energy $E_1 - E_2$, must also have frequency $\nu = \frac{E_1 - E_2}{h}$. In order to create such a frequency, the atom itself must have charges, etc, which are vibrating with that frequency. This vibration must have something to do with the states of the system in changing between the two energy levels.

In our two dimensional example, with the energy matrix (Hamiltonian) given by

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \tag{1}$$

the matrix S for example is of a form which links the two levels. If we multiply the vector corresponding to the upper level by S we get the lower, and vice-versa. If this S corresponds to what it is that is causing the jump and the emission of radiation, we would expect that S should be "vibrating" or oscillating. He hypothesised that $\langle E_2|S|E_1\rangle$ should correspond to whatever it is that produces the radiation, and that this should oscillate like $e^{i2\pi\frac{(E_1-E_2)}{h}t}$. In order that the matrix S be hermitean, we must also have that $\langle E_1|S|E_2\rangle = (\langle E_2|S|E_1\rangle)^\dagger$ should go as $e^{-i2\pi\frac{(E_1-E_2)}{h}t}$. Ie, $S(t)$ should look like

$$S(t) = \begin{pmatrix} 0 & e^{-i\frac{2\pi(E_1-E_2)}{h}t} \\ e^{i\frac{2\pi(E_1-E_2)}{h}t} & 0 \end{pmatrix} \tag{2}$$

Because the expression $\frac{h}{2\pi}$ occurs so often, Dirac gave it the name $\hbar$, pronounced h-bar. We have

$$\frac{dS(t)}{dt} = \frac{1}{\hbar}\begin{pmatrix} 0 & -i(E_1-E_2)e^{-i\frac{E_1-E_2}{\hbar}t} \\ i(E_1-E_2)e^{i\frac{(E_1-E_2)}{\hbar}t} & 0 \end{pmatrix} \tag{3}$$

This looks a bit of a mess. Is there some way we can compactify this expression? Let us look at $S(t)H$ and $HS(t)$, the product of the two matrices. We find

$$\begin{aligned} S(t)H &= \begin{pmatrix} 0 & E_2 e^{-i\frac{E_1-E_2}{\hbar}t} \\ E_1 e^{i\frac{E_1-E_2}{\hbar}t} & 0 \end{pmatrix} \\ HS(t) &= \begin{pmatrix} 0 & E_1 e^{-i\frac{E_1-E_2}{\hbar}t} \\ E_2 e^{i\frac{E_1-E_2}{\hbar}t} & 0 \end{pmatrix} \end{aligned} \tag{4}$$

and

$$S(t)H - HS(t) = \begin{pmatrix} 0 & (E_2 - E_1)e^{-i\frac{E_1-E_2}{\hbar}t} \\ (E_1 - E_2)e^{i\frac{E_1-E_2}{\hbar}t} & 0 \end{pmatrix} = i\hbar\frac{dS(t)}{dt} \tag{5}$$

While in our case we only have two energy levels, Heisenberg worked with systems with many energy levels, and exactly the same expression applies. Ie, for an arbitrary attribute $A(t)$ of a system and with an energy matrix H, the time derivative of A is given by

$$i\hbar\frac{dA(t)}{dt} = A(t)H - HA(t) \tag{6}$$

The expression on the right hand side occurs so often in quantum mechanics that it is given a special symbol. If A and B are two matrices, then the combination

$$AB - BA \equiv [A, B] \tag{7}$$

is called the commutator. It is a measure of how much A and B do not commute when multiplied together.

The Heisenberg equation of motion then tells us that the time derivative of any attribute arises purely out of its not commuting with the energy/Hamiltonian.

$$i\hbar\frac{dA}{dt} = [A, H] \tag{8}$$

is the key equation of the dynamics of any quantum system.