Physics 200-04
Complex numbers

Complex Numbers

One of the themes throughout the history of mathematics was the expansion of what one regarded as numbers. Mathematics began with counting– the positive integers. It was gradually realised that the process of subtraction could be unified with that of addition by the invention of negative integers as well as the special number 0. (It actually took a long time for zero to be discovered).

Then, the idea of fractions– divisions of objects and thus of numbers into equal pieces. In particular, fractions, the rational numbers, were the solutions of the simplest of linear equations with integer coeficients. Eg, $3x - 7 = 0$. We note that we create an equation using just integers, and the solution requires something other than integers.

One could limit oneself to finding only integer solutions (in general possible even for linear equations only if the equations underdetermine the solutions-eg, Diophthantene solutions), or try to expand one's notion of numbers to encompase a class which would be a solution to equations like that. Chosing the latter road, one came up with the concept of the rational numbers (integers divided by integers).

The next discovery, that of the irrationals, came as a huge shock to the philosophers. The proof that the solution to the equation $x^2 = 2$ could not be expressed as any rational number caused no end of trouble, From < *http://www.peacelink.de/keyword/Irrational_number.php* >

"The discovery of irrational numbers is usually attributed to Pythagoras or one of his followers, who produced a (most likely geometrical) proof of the irrationality of the square root of 2. One story is that one of his followers called Hippasus discovered irrational numbers when trying to represent square root of 2 as a fraction. However Pythagoras believed in the absoluteness of numbers, and could not accept the existence of irrational numbers. He could not disprove their existence through logic, but his beliefs would not accept the existence of irrational numbers and so he sentenced Hippasus to death by drowning."

Mathematics was a serious subject in those days!

Again one could accept that such numbers simply did not exist (but that would be embarassing since the hypoteneuse of a right triangle with two sides equal to one is given by that value, and outlawing such numbers would mean that the hypoteneuse has no length) or again extend the notion of what a number is to include the irrationals. Going the second route, one could now solve may equations in terms of the rational and irrational numbers.

However there were still equations which could not be solved. The simplest of these is

$$x^2 + 1 = 0 \tag{1}$$

Just as $x^2 = 2$ has no solutions in the rationals it is easy to see that $x^2+1 = 0$ has no solution for any real number (rational or irrational) x since all squares of rationals and irrationals are positive numbers.

It turned out that one could again extend one's definition of what one meant by a number by simply defining a solution to this equation and giving this solution a name. In the case of the $\sqrt{2}$ in some sense the extention to irrationals was not very far. While there is no rational which is exactly equal to this number, one can find rationals whose square is arbitrarily close to $\sqrt{2}$. However there is no number, rational or irrational whose square is at all near -1.

We will thus simply augment our idea of number by adding this number, the solution of $x^2 = -1$ to what we think of as a number. This number is traditionally called i although in the engineering literature it is often called j.

Given this one new number, we can now extend our idea of numbers by multiplying this number– so that say $7i$ will be the number whose square is -49. We can moreover imagine also adding such numbers together– to get something say like $23 + 2i$. Of course at present this is just a formal sum. It is not clear what it means. But we can extend it by giving such numbers more properties.

Complex rules:

A complex number is an object which is written in the form $a + ib$ where bi means b multiplied by i– ie the number whose square is $-b^2$. We give it a number of properties.

0) Equality. If $a+ib$ and $c+id$ are two complex numbers, then these two numbers are equal only if $a = c$ and $b = d$. Both must be true for the two to be equal.

i) Addition. If $a+ib$ and $c+id$ are two complex numbers, where a, b, c, d

are all real numbers, then

$$(a+ib)+(c+id)=(a+c)+i(b+d) \tag{2}$$

It is clear from this definition of addition that $(a+ib)+(c+id)=(c+id)+(a+ib)$. Ie the order of addition does not matter.

ii) Multiplication: We can define multiplication by treating i as if it is some algebraic variable. Thus

$$(a+ib)(c+id)=ac+iad+ibc+(i^2bd)=(ac-bd)+i(ad+bc) \tag{3}$$

where we have in addition used $i^2=-1$. Ie, multiplication proceeds just as if i were some variable x, except that whenever you see an i^2 you can replace it with -1.

iii) Division: $(a+ib)/(c+id)$ is the complex number which, when multiplied by $c+id$ is equal to $a+ib$. Amazingly not only is there an answer, there is a unique answer. We start by noting that the simplest division $(c-id)/(c-id)=1$. Thus we can multiply $(a+ib)/(c+id)$ by this expression for unity (and by the multiplication law, this is the same as the original number). Now, we also assume that

$$((a+ib/c+id))((e+if)/(g+ih))=((a+ib)(e+if))/((c+id)(g+ih)) \tag{4}$$

This leads to

$$\begin{aligned}((a+ib)/(c+id)) &= ((a+ib)/(c+id))((c-id)/(c-id)) \\ &= ((a+ib)(c-id))/((c+id)(c-id)) \\ &= ((a+ib)(c-id))/(c^2+d^2) \\ &= ((ac+bd)/(c^2+d^2))+i((bc-ad)/(c^2+d^2)) \end{aligned} \tag{5}$$

Ie we can explicitly define division of complex numbers by numbers. complex multiplication and division by real numbers.

iv) Complex Conjugation: As in the above it is often useful to define another complex number related to some number $c+id$ by simply reversing the sign of the i. This is called a complex conjugate of the original number. It is designated by a superscript star.

$$(c+id)^*=(c-id) \tag{6}$$

where c and d are real numbers. If x say is supposed to represent a complex number, then one writes x^* without being able to evaluate it. Note that

$$(c+id)(c+id)^* = c^2 + d^2 \tag{7}$$

is a real positive number. It is called the square of the modulus of the complex number $c + id$. This is often written as

$$|c+id|^2 = (c+id)(c+id)^* = c^2 + d^2 \tag{8}$$

or

$$|c+id| = \sqrt{c^2 + d^2} \tag{9}$$

where one always takes the positive square root.

An astonishing feature of complex numbers is that they complete the chain of finding solutions to equations. In each of the previous cases one could find equations expressed purely in terms of one class of numbers such that those equations had no solutions in terms of that kind of number. Now however, any polynomial with complex coefficients always has solutions which are complex numbers. One does not have to invent some new kind of numbers. The complex numbers complete the kinds of numbers one needs to handle equations.

The complex numbers have all of the features of the rationals and reals as well. Ie, one has addition, subtraction, multiplication, division, an idea of both 0 and 1. (There exist other types of numbers, quaternions and octonians which share with complex, real and rational numbers these features, but they have found very very little application).

examples

$$\begin{aligned}
(7-5i)(6+i) &= 42 + 7i - 30i - 5i^2 = 42 + 5 - 23i = 45 - 23i \\
(2.113 - .0776i)^* &= (2.113 + .0776i) \\
(3+4i)/(2+5i) &= (3+4i)(2-5i)/((2+5i)(2-5i)) \\
&= (26-7i)/29 = (26/29) - (7/29)i
\end{aligned} \tag{10}$$

Complex Matrices

Just as we have dealt with matricees all of whose elements are real numbers we can deal with matrices whose elements are all complex numbers instead. The law of multiplication of these is exactly the same as the multiplication of matrices with real coeficients, as far as the elements are concerned. Thus

$$\begin{aligned}\begin{pmatrix}1+i & 2+3i \\ 0 & -2+i\end{pmatrix} \quad & \begin{pmatrix}1+i & 1+i \\ 2+2i & 4\end{pmatrix} \\ &= \begin{pmatrix}(1+i)^2+(2+3i)(2+2i) & (1+i)(1+i)+(2+3i)(4) \\ 0(1+i)+(-2+i)(2+i) & 0(1+i)+(-2+i)4\end{pmatrix} \\ &= \begin{pmatrix}-2+12i & 8+14i \\ -5 & -8+4i\end{pmatrix}\end{aligned} \tag{11}$$

Ie, although complex matrices are more complex than real ones, they are no more complicated.

There are two more ideas which are important for complex matrices. The first is complex conjugation. If A is a complex matrix, the the matrix A^* is defined to be the matrix formed by taking the complex conjugate of each element of A. Thus

$$\begin{pmatrix}-2+12i & 8+14i \\ -2+i & -8+4i\end{pmatrix}^* = \begin{pmatrix}-2-12i & 8-14i \\ -2-i & -8-4i\end{pmatrix} \tag{12}$$

The transpose is the same as for ordinary matrices but is not used that often. Instead there is another operations, called the Dirac adjoint, and is symbolised by a dagger.

$$A^\dagger = (A^*)^T \tag{13}$$

It is the matrix obtained by first taking the complex conjugation of the matrix, and then taking the transpose. Thus

$$\begin{pmatrix}-2+12i & 8+14i \\ -2+i & -8+4i\end{pmatrix}^\dagger = \begin{pmatrix}-2-12i & -2-i \\ 8-14i & -8-4i\end{pmatrix} \tag{14}$$

A Hermitean matrix is one whose Dirac adjoint is the matrix itself.

$$H^\dagger = H \tag{15}$$

This must mean that all elements along the diagonal must be real, while across the diagonal the elements must be complex conjugates of each other.

It will turn out that Hermitean matrices play a crucial role in quantum mechanics.

Eigenvalues and Eigenvectors

If A is a square (nxn) complex matrix, then the column matrix V (nx1)– usually called a vector with one column and n rows is called an eigenvector (eigen means "itself" in German) if the product of A with the vector V – ie AV– which is again an $nx1$ matrix– is a multiple of V. Ie, V is an eigenvector of A if and only if V obeys

$$AV = \lambda V \tag{16}$$

for some complex number λ. The Vector V which obeys this equation is called an eigenvector of A while the number λ is called the eigenvalue of this eigenvector.

For example, consider the matrix

$$A = \begin{pmatrix} 1-i & 2+3i \\ 0 & -2+i \end{pmatrix} \tag{17}$$

An eigenvector of this matrix is

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{18}$$

since

$$AV_1 = \begin{pmatrix} 1-i & 2+3i \\ 0 & -2+i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-i \\ 0 \end{pmatrix} = (1-i) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{19}$$

(Recall that when you multiply a matrix by a constant, you multiply every element of the matrix by that constant). The eigenvalue associated with this eigenvector is $\lambda_1 = 1 - i$.

Similarly, the vector

$$V_2 = \begin{pmatrix} 2+3i \\ -3+2i \end{pmatrix} \tag{20}$$

is another eigenvector of this matrix with eigenvalue $\lambda_2 = -2 + i$.

(In this particular case the eigenvalues are also just the elements of the diagonal of the matrix, but this is NOT true in general)

The Hermitean matrices are important in quantum mechanics because the eigenvalues are always real numbers, not complex numbers. (I will denstrate this later in the course.)