Physics 200-05

Assignment 8

1. Consider $H = \frac{1}{2}\vec{h}\cdot\vec{\sigma}$ as the energy matrix. and that $\vec{n}_\psi\cdot\vec{\sigma}$ with $\vec{n}_\psi\cdot\vec{n}_\psi = 1$, is the matrix for which $|\psi\rangle$ is the eigenvector for the eigenvalue +1. It can be shown that

$$\vec{n}_\psi = \langle\psi|\vec{\sigma}|\psi\rangle \tag{1}$$

Show that the equation of motion for $\vec{n}_\psi$ is given by

$$\frac{d\vec{n}}{dt} = \frac{1}{\hbar}\vec{h}\times\vec{n} \tag{2}$$

$\vec{n}$ is defined as the expectation value of the three attributes $\sigma_1, \sigma_2,\ \sigma_3$ in the state $|\psi\rangle$. Since under the Schroedinger representation, it is $|\psi\rangle$ that changes with time, it is due to that change that $\vec{n}$ changes. From the lectures, if A is an attribute, then

$$\frac{d\langle\psi|A|\psi\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[H,A]|\psi\rangle$$

.

Thus we have

$$\begin{aligned} d\langle\psi|\sigma_i|\psi\rangle &= \frac{i}{\hbar}\langle\psi|[\frac{1}{2}\vec{h}\cdot\vec{\sigma},\sigma_i]|\psi\rangle \\ &= \frac{i}{\hbar}\langle\psi|[\frac{1}{2}(h_1\sigma_1+h_2\sigma_2+h_3\sigma_3),\sigma_i]|\psi\rangle \end{aligned} \tag{3}$$

Now we use the multiplication of the sigma matrices.

$$\begin{aligned} \sigma_1\sigma_2 &= -\sigma_2\sigma_1 = i\sigma_3 \\ \sigma_2\sigma_3 &= -\sigma_3\sigma_2 = i\sigma_1 \\ \sigma_3\sigma_1 &= -\sigma_1\sigma_3 = i\sigma_2 \end{aligned} \tag{4}$$

and

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \tag{5}$$

Thus we have

$$
\begin{aligned}
[\sigma_i, \sigma_i] &= 0 \\
[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = -[\sigma_2, \sigma_1] &= 2i\sigma_3 \\
[\sigma_2, \sigma_3] = -[\sigma_3, \sigma_2] &= 2i\sigma_1 \\
[\sigma_3, \sigma_1] = -[\sigma_1, \sigma_3] &= 2i\sigma_2
\end{aligned}
\tag{6}
$$

Thus

$$
\begin{aligned}
\frac{dn_1}{dt} &= \frac{1}{\hbar}\langle\psi|\frac{i}{2}(h_1[\sigma_1, \sigma_1] + h_2[\sigma_2, \sigma_1] + h_3[\sigma_3, \sigma_1])|\psi\rangle \\
&= \frac{1}{\hbar}\langle\psi|\frac{1}{2}(h_1 0 + h_2(-2i\sigma 3) + h_3(2i\sigma_2))|\psi\rangle = \frac{1}{\hbar}(h2\langle\psi|\sigma_3|\psi\rangle - h3\langle\psi|\sigma_2|\psi\rangle) \\
&= \frac{1}{\hbar}(h_2 n_3 - h_3 n_2)
\end{aligned}
\tag{7}
$$

By exactly the same argument

$$
\begin{aligned}
\frac{dn_2}{dt} &= \frac{1}{\hbar}(h_3 n_1 - h_1 n_3) \\
\frac{dn_3}{dt} &= \frac{1}{\hbar}(h_1 n_2 - h_2 n_1)
\end{aligned}
\tag{8}
$$

But the right hand side is precisely the components of the cross product.
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Let us assume that initially $n_{\psi 3} = 1$ and $n_{\psi 1} = n_{\psi 2} = 0$. and that $h_3 = \epsilon\cos(\theta)$, $h_1 = \epsilon\sin(\theta)$, $h_2 = 0$.

Then as a function of time show that

$$
\begin{aligned}
n_3 &= (\cos(\theta)^2) + \cos(\omega t)\sin(\theta)^2 \\
n_1 &= \sin(\theta)\cos(\theta)(1 - \cos(\omega t)) \\
n_2 &= -\sin(\theta)\sin(\omega t)
\end{aligned}
\tag{9}
$$

where $\omega = \frac{1}{\hbar}\epsilon$.

By far the easiest way to do this is just to insert the expression into the differential equation and test it to see if it works. Thus

$$
\begin{aligned}
\frac{dn_1}{dt} &= \omega\sin(\theta)\cos(\theta)(\sin(\omega t)) \\
(\vec{h} \times \vec{n})_1 &= h_2 n_3 - h_3 n_2 = 0 - \epsilon\cos(\theta)(-\sin(\theta)\cos(\theta)(\sin(\omega t)))
\end{aligned}
\tag{10}
$$

and the two sides are equal to each other since $\omega = \frac{\epsilon}{\hbar}$.

Furthermore at $t = 0$, $n3 = (\cos(\theta)^2) + \cos(\omega t)\sin(\theta)^2 = 1$, $n_2 = 0$ and $n_1 = \sin(\theta)\cos(\theta)(1-1) = 0$ as required. (Note that anytime if you are given an answer a way of showing it is the answer is to plug the answer into the equations and show it satisfies them.)

You could also solve the equation directly.

$$\begin{aligned}
\frac{dn_1}{dt} &= \frac{1}{\hbar}(h_2 n_3 - h_3 n_2) = -\omega\cos(\theta)n_2 \\
\frac{dn_2}{dt} &= \omega(\cos(\theta)n_1 - \sin(\theta)n_3) \\
\frac{dn_3}{dt} &= \omega(\sin(\theta)n_2
\end{aligned} \tag{11}$$

Let $X = \cos(\theta)n_1 - \sin(\theta)n_3$. Then

$$\begin{aligned}
\frac{dn_2}{dt} &= \omega X \\
\frac{dX}{dt} &= -\omega n2
\end{aligned} \tag{12}$$

Solving the second equation for n_2 and substituting into the first, we have

$$\frac{d^2X}{dt} = -\omega^2 X$$

This has as solutions $X = a\cos(\omega t) + b\sin(\omega t)$ where a and b are some constants. Then $n_2 = -\frac{1}{\omega}\frac{dX}{dt} = b\cos(\omega t) - a\sin(\omega t)$

Now at $t = 0$, $n_2 = 0$, so b must be zero. Similarly at $t = 0$, $X = -\sin(\theta)$ so $a = -\sin(\theta)$.

This is not quite enough to solve the problem. However, if we define $Y = \cos(\theta)n_3 + \sin(\theta)n_1$, then $\frac{dY}{dt} = 0$, and at $t = 0$, $Y = \cos(\theta)$. Solving from X, Y for n_1 and n_3 we find exactly the claimed solution.

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σ_3 is the attribute which represents the location of the N atom in ammonia, the probability of tunneling from the right (+1) to the left side (-1) becomes small as the difference in expectation value of energy for the particle on the left or right differs by more than the "tunneling energy" ($\epsilon\sin(\theta)$).

Show that if $\theta = \frac{\pi}{2}$, there is a complete tranfer of probability from the right to left (+1 to -1 eigenvalue). As θ goes to zero, the probability of finding the particle on the left at any time goes to zero.

The eigenvector for the particle being on the left is $|left\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the expectation value of σ_3 is n_3. But that expectation value is $P_{right}1 + P_{left}(-1)$. Thus the probability of being on the left is (using that $P_{right} + P_{left} = 1$) $P_{left} = \frac{1}{2}(1 - n_3)$. Thus, the probability of finding it on the left is

$$P_{left} = \frac{1}{2}(1 - (\cos(\theta)^2 - \sin(\theta)^2\cos(\omega t)) = \frac{1}{2}\sin(\theta)^2(1 - \cos(\omega t)) \qquad (13)$$

This goes from 0 at $t = 2r * \pi/\omega$ to $\sin(\theta)^2$ at $t = (2r+1)\pi/\omega$ where r is an integer. Thus if $\theta = \pi/2$, the probability of find the atom on the left goes to 1 at selected times. If θ is small however, the probability of finding on the left is never large.

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(Note that this is an experiment being done by Walter Hardy and his group in the basement of Hennings right now, where instead of N in ammonia is the orientation of a giant molecule containing iron. Their big problem is to ensure that the angle θ is as near $\pi/2$ as possible)

2. Show that an alternative way of solving the time dependent equations of motion of the two level system is by directly solving

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}H|\psi\rangle \qquad (14)$$

Assume that $H = \frac{\epsilon}{2}\sigma_2$, and that $|\psi\rangle = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$ with $\psi_1(0) = 1$ and $\psi_2(0) = 0$. Find ψ_1 and ψ_2 as a function of time.

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}H|\psi\rangle$$

$$\begin{pmatrix} \frac{d\psi_1}{dt} \\ \frac{d\psi_2}{dt} \end{pmatrix} = -i\frac{\epsilon}{2\hbar}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\epsilon}{2\hbar}\begin{pmatrix} -\psi_2 \\ \psi_1 \end{pmatrix} \qquad (15)$$

Thus we have the two equations to solve

$$\frac{d\psi_1}{dt} = -\frac{\epsilon}{2\hbar}\psi_2$$
$$\frac{d\psi_2}{dt} = \frac{\epsilon}{2\hbar}\psi_1 \qquad (16)$$

Solving the second for ψ_1 and substituting into the first, we get

$$\frac{d^2\psi_2}{dt^2} = -(\frac{\epsilon}{2\hbar})^2\psi_2 \tag{17}$$

with solutions $\psi_2 = b\cos(\omega t) - a\sin(\omega t)$ and$\psi_1 = -a\cos(\omega t) - b\sin(\omega t)$ where $\omega = \frac{\epsilon}{2\hbar}$.

Note that if we want the vector normalised, we want $\langle\psi||\psi\rangle = 1$ or

$$\begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = a^*a + b^*b = 1 \tag{18}$$

Note that if the vector is normalised at any one instant of time, it is normalised at all instants of time.

Since $\psi_1(0) = 1$ and $\psi_2(0) = 0$, we have $a = -1$ and $b = 0$, so that $\psi_1 = \cos(\omega t)$, $\psi_2 = \sin(\omega t)$

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3. Consider the state for the two two-level states given by

$$\begin{aligned} |\psi\rangle &= N_\psi\left(|+1;1\rangle \otimes |-1;1\rangle + |-1;1\rangle \otimes |+1;1\rangle\right) \\ |\phi\rangle &= N_\phi\left(|+1;1\rangle \otimes |+1;1\rangle + |-1;1\rangle \otimes |-1;1\rangle\right) \end{aligned} \tag{19}$$

where $|+1;1\rangle$ means the eigenstate with the $+1$ eigenvalue for the attribute Σ_1 in the case of the first particle and $+1$ for Ξ_1 for the second. Ie, the first value is the eigenvector, and the second is the sigma matrix of which this is the eigenvector. Σ_i are the three sigma matrices for the first particle, and Ξ_i are the three sigma matrices for the second particle. What are possible values for the normalisation factors N_ψ and N_ϕ? What is the inner product $\langle\phi||\psi\rangle$.

(Do not try to expand out the direct products in terms of matrices.)

$$\begin{aligned} \langle\psi||\psi\rangle &= N_\psi^*\left(\langle +1;1| \otimes \langle -1;1| + \langle -1;1| \otimes \langle +1;1|\right) \\ &\qquad N_\psi\left(|+1;1\rangle \otimes |-1;1\rangle + |-1;1\rangle \otimes |+1;1\rangle\right) \\ &= N_\psi^* N_\psi\left((\langle +1;1| \otimes \langle -1;1|)(|+1;1\rangle \otimes |-1;1\rangle)\right. \\ &\qquad +(\langle +1;1| \otimes \langle -1;1|)(|-1;1\rangle \otimes |+1;1\rangle) \\ &\qquad +(\langle -1;1| \otimes \langle +1;1|)(|+1;1\rangle \otimes |-1;1\rangle) \\ &\qquad \left.+(\langle -1;1| \otimes \langle +1;1|)(|-1;1\rangle \otimes |+1;1\rangle)\right) \end{aligned}$$

$$
\begin{aligned}
&= N_\psi^* N_\psi \left((\langle +1;1|| +1;1\rangle) \otimes (\langle -1;1|| -1;1\rangle) \right.\\
&\quad +(\langle +1;1|| -1;1\rangle) \otimes (\langle -1;1|| +1;1\rangle)\\
&\quad +(\langle -1;1|| +1;1\rangle) \otimes (\langle +1;1|| -1;1\rangle)\\
&\quad \left. +(\langle -1;1|| -1;1\rangle) \otimes (\langle +1;1|| +1;1\rangle) \right)\\
&= N_\psi^* N_\psi (1 \otimes 1 + 0 \otimes 0 + \langle -1;1|| +1;1\rangle \otimes 0 + 1 \otimes 1) = 2N_\psi^* N_\psi
\end{aligned} \tag{20}
$$

Thus $N_\psi^* N_\psi = 1/2$ and $N_\psi = \frac{1}{\sqrt{2}}$.

Similarly for ϕ.

Finally

$$
\begin{aligned}
\langle\psi||\phi\rangle &= \frac{1}{2}((\langle +1;1| \otimes \langle -1;1| + \langle -1;1| \otimes \langle +1;1|)\\
&\quad (| +1;1\rangle \otimes | +1;1\rangle + | -1;1\rangle \otimes | -1;1\rangle)\\
&= 0
\end{aligned} \tag{21}
$$

since in each term of the product one of the bras is orthogonal to one of the kets in the direct product.

4. If Σ_1 is the Pauli spin matrix for the first particle with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and Ξ_1 is the Pauli spin matrix for the second particle with the same matric $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, show in problem 3 that both $|\psi\rangle$ and $|\phi\rangle$ are eigenvectors of the matrix $\Sigma_1 \otimes \Xi_1$. What are the eigenvalues?

$$
\begin{aligned}
\Sigma_1 \otimes \Xi_1 \ & (| +1;1\rangle \otimes | -1;1\rangle + | -1;1\rangle \otimes | +1;1\rangle)\\
&= (\Sigma_1| +1;1\rangle) \otimes (\Xi_1| -1;1\rangle) + \Sigma_1| -1;1\rangle \otimes \Xi_1| +1;1\rangle\\
&= (1| +1;1\rangle \otimes (-1)| -1;1\rangle + (-1)| -1;1\rangle \otimes (+1)| +1;1\rangle)\\
&= -(| +1;1\rangle \otimes | -1;1\rangle + | -1;1\rangle \otimes | +1;1\rangle)
\end{aligned} \tag{22}
$$

This is clearly an eigenvector with eigenvalue -1.

$$
\begin{aligned}
\Sigma_1 \otimes \Xi_1 \ & (| +1;1\rangle \otimes | +1;1\rangle + | -1;1\rangle \otimes | -1;1\rangle)\\
&= (\Sigma_1| +1;1\rangle \otimes \Xi_1| +1;1\rangle + \Sigma_1| -1;1\rangle \otimes \Xi_1| -1;1\rangle)\\
&= ((+1)| +1;1\rangle \otimes (+1)| +1;1\rangle + (-1)| -1;1\rangle \otimes (-1)| -1;1\rangle)\\
&= (| +1;1\rangle \otimes | +1;1\rangle + | -1;1\rangle \otimes | -1;1\rangle)
\end{aligned} \tag{23}
$$

which is the eigenvector with eigenvalue +1.

Ie, $|\psi\rangle$ is an eigenvector with eigenvalue +1 and $|\phi\rangle$ is and eigenvector with eigenvalue -1.

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5. Show expicitly that if

$$|1;3\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$
$$|-1;3\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{24}$$

for each of the two particles, then the four dimensional vector

$$|1;3\rangle \otimes |1;3\rangle + |-1;3\rangle \otimes |-1;3\rangle \tag{25}$$

cannot be writen as a simple product

$$|\chi\rangle \otimes |\xi\rangle \tag{26}$$

for any choice of $|\chi\rangle$ and $|\xi\rangle$. Such a vector for two particles which cannot be written as a simple product of vectors for the two single particles is called an entangled state.

$$|\chi\rangle \otimes |\xi\rangle = \begin{pmatrix}\chi_1\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\\\chi_2\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\chi_1\xi_1\\\chi_1\xi_2\\\chi_2\xi_1\\\chi_2\xi_2\end{pmatrix} \tag{27}$$

Note the ratio of the first component of the matrix over the second is the same as the ratio of the third over the fourth.

But

$$|1;3\rangle \otimes |1;3\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{28}$$

and

$$|-1;3\rangle \otimes |-1;3\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{29}$$

And the sum is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

. The ratio of the first by the second is infinity while that of the third by the fourth is 0, which are not equal to each other. Thus this sum cannot be written as the direct product of two other matrices.

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(In this case do expand the direct products of ket vectors in terms of matrices.)

[Note: in the physics literature, that $\otimes$ symbol is almost always omitted. Thus $|\chi\rangle \otimes |\xi\rangle$ is written as $|\chi\rangle|\xi\rangle$ and you are expected to know that it is the direct product that is being used if you are refering to two separate particles. Similarly $\Sigma_1 \otimes \Xi_1$ is written as $\Sigma_1\Xi_1$ where you are to remember that this is a direct product not a matrix product because Σ_1 and Ξ_1 belong to two separate particles. It is never correct to matrix multiply attributes which belong to separate particles.]