Physics 200-04
Assignment 3

1.Doppler shift: A light flashes once a second (according to its own clock). It is traveling with velocity .9c with respect to an observer, in the direction of the positive x direction and at a distance of 1 light day away along the y direction. What is the frequency of the flashes as seen by the observer as a function of time. What is the limiting frequency when the x value is very large and negative, when $x = 0$ and when x is large and positive.

Note: You need to take both the time dilation and the time it takes light to travel the distance into account.

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[4]

{ 2 for realising that there are two effects here. One is for the change in the frequency due to the time dilation special relativistic effect, and 1 for realising that the fact that the distance from the moving light to the origin changes, which will also alter the time between ticks as received at the origin. 1 for calculating the latter correctly , and finally 1 for getting the answer right. }

If the light is emitted at time t, the time it takes for the light to reach the origin is $\sqrt{y^2 + (vt)^2}$ and will thus arrive at the origin at the time $\tau = t + \sqrt{y^2 + (vt)^2}$ The time as measured in the moving objects frame is $\tilde{t} = \sqrt{1 - v^2}t$ Thus the relation between the time at which the light arrives at the origin and the time at which it was emitted is

$$\tau = \frac{1}{\sqrt{1 - v^2}}\tilde{t} + \sqrt{y^2 + \frac{v^2}{1 - v^2}\tilde{t}^2} \tag{1}$$

The time between the reception of the pulses is

$$\Delta\tau = \frac{1}{\sqrt{1 - v^2}}\Delta\tilde{t} + \frac{1}{\sqrt{y^2 + \frac{v^2}{1-v^2}\tilde{t}^2}}\frac{v^2}{1 - v^2}\tilde{t}\Delta\tilde{t} \tag{2}$$

For large negative time $\tilde{t}$, the y^2 term in the square root is negligible and

$$\Delta\tau = \frac{1 - v}{\sqrt{1 - v^2}}\Delta\tilde{t} = \sqrt{\frac{1 - v}{1 + v}} \tag{3}$$

Ie, the received frequency is higher than the frequency of the source.

At a time $\tilde{t} = 0$ when the moving object is moving transversly to the observer, the expression becomes

$$\Delta\tau = \frac{\Delta t}{\sqrt{1 - v^2}} \tag{4}$$

This is called the transverse doppler shift. The shift in the pulse frequency is purely that due to the different rates at which the moving and stationary clock tick. Finally if the clock is at very large positive time $\tilde{t}$, the expression is

$$\Delta\tau = \sqrt{\frac{1+v}{1-v}}\Delta\tilde{t} \tag{5}$$

which is larger than the ticks of the source.

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2. A distant star, distance 10^4 light years away, ejects a blob of hot matter at a velocity 12/13 c almost directly toward the observer. The angle that the velocity of blob makes with the direction to the observer is 20 degrees. Looking in the sky at the image of the blob separating from the image of the star, with what velocity would the observer see the blob moving? (Note that taking into account the time it takes light to reach the observer is important.)

Note: when this was first seen by astronomers it caused a huge sensation until the community was convinced about what was happening and that this did not violate special relativity.

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[5] The key to this problem is to realise that light takes a finite time to get to the earth, and since the distance between the star and earth decreases with time, the time it takes the light to get to the earth also decreases{1}.

If the star is a distance L away, then the path of the blob is

$$X_{blob} = L - vcos(theta)t \tag{6}$$

$$Y(blob) = vsin(theta)t \tag{7}$$

If the light is emitted at time t, the time that light gets to the earth is

$$t_{earth} = t + \frac{X_{blob}}{c} \tag{8}$$

{1} (Since the blob is so far away, the change in distance due to the motion of the blob in the y direction is completely negligible)

Thus, at time t_{earth} the blob will be seen to be at position $X_{blob}(t)$, $Y_{blob}(t)$. solving for t as a function of t_{earth}, the we have

$$t = \frac{t_{earth} - \frac{L}{c}}{1 - \frac{v}{c}cos(\theta)} \tag{9}$$

Thus the position of the blob as seen from the earth expressed in the earth time t_{earth} is

$$X_{blob} = L - vcos(\theta)\frac{t_{earth} - \frac{L}{c}}{1 - \frac{v}{c}cos(\theta)}) \tag{10}$$

$$Y_{blob} = vsin(\theta)\frac{t_{earth} - \frac{L}{c}}{1 - \frac{v}{c}cos(\theta)}) \tag{11}$$

{1} Thus the y component of the velocity of that blob as seen from earth where the velocity is measured in earth time is

$$v_{y\ apparent} = \frac{v sin(\theta)}{1 - \frac{v}{c} cos(\theta)} \quad (12)$$

in fact the x velocity will be impossible to see since it does not change the position of blob against the sky.

Now, since $\frac{v}{c}$=.9, we have

$$v_{y\ apparent} = v \frac{sin(\theta)}{1 - .9 cos(\theta)} \quad (13)$$

{1}

At 20 degrees the transverse velocity is apparently about 2c. {1}

As stated this confused astronomers at first. One of the early such galaxies had an apparent velocity of about 10c (Its true velocity was about .99c). Ie, they measured the change in angle between the blob and the star as a function of time, multiplied this by the distance to the star and found an apparent transverse velocity much greater than c. It took them a little while to realise that they had to take into account the fact that the light had a shorter and shorter distance to travel to get to the observr.

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3. A structure as in figure 1 is such that the projection of the structure A would just touch the detonator when the flanges of figure A just touch the bars of Figure B. Now A is shot at B at a sizable fraction of the speed of light. In the frame of figure B, A is shortened and thus the flanges will touch the bars of A before the projection touches the detonator, and will stop A before the projection hits the detonator. In A's frame however, B is shorter, and the projection of A hits the detonator before the flanges touch B and the bomb is detonated.

Is the bomb detonated or not? What is wrong with the argument that predicts the opposite?

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[4]

{1}This is precisely the same as the pole in the barn problem. We see that depending on which frame we look at the problem in, either the front of the T hits the detonator before the flanges hit the ends of the U, or the flanges hit first. But this means that the flanges hitting and the end of the T hitting the detonator are {1} spacelike separated. { 1 }Thus no signal can travel from one of those events to the other. In the frame of the U, the flanges hit first. But no signal can be sent from teh flanges to the end of the T, and thus nothing can stop the end of the T before it hits the detonator. In this frame the flanges will hit the ends of the U, and a shock wave will travel down the T, but the end of the T will keep moving and hit the detonator long before the shock wave can reach the end of the T. The T cannot stop instantly.

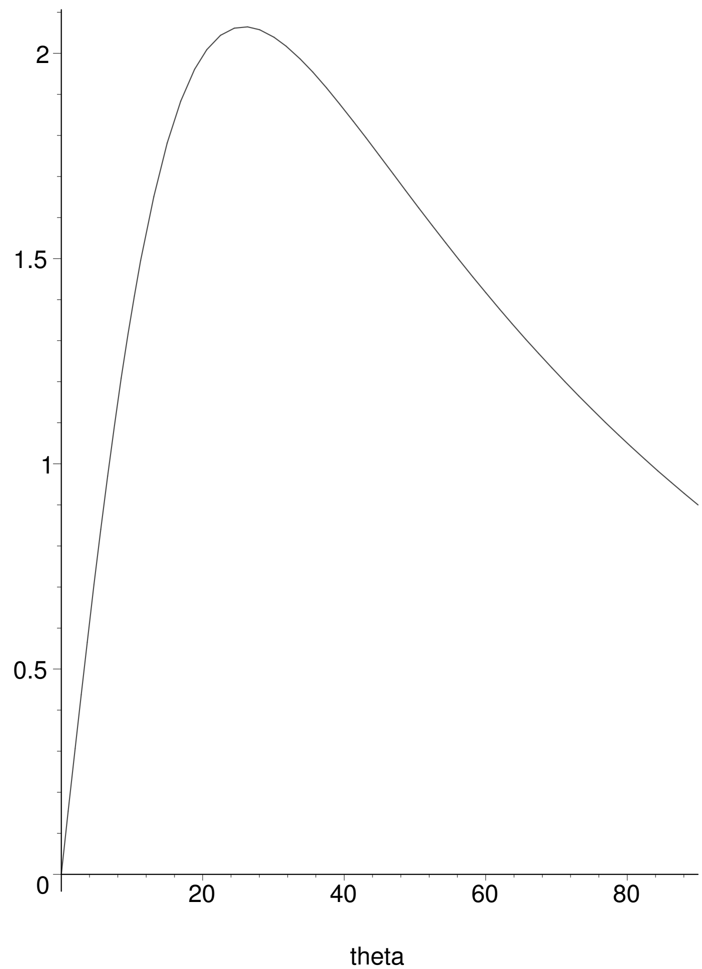

Figure 1: Plot of the transverse velocity seen on earth as a function of the angle in degrees

In the frame of the T, the bottom hits the detonator first and again no signal can travel back to the flanges in time to stop them from hitting.

The detonator will go off.{1}

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4. Thomas Precession: Consider a particle moving with velocity v in the x direction from the point of view of some observer. In the frame of the particle, the particle now picks up a small velocity δv in the y direction.

a) Show that the matrix representing the Lorents transformation from the observers frame to the particle;'s old frame is

$$L_v = \begin{pmatrix} \cosh(\mu) & -\frac{v}{c}\cosh(\mu) & 0 & 0 \\ -\frac{v}{c}\cosh(\mu) & \cosh(\mu) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\cosh(\mu) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, while the transformation from the particle's old frame to the particle's new frame, to linear order in δv is

$$L_\delta = \begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\delta v}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that the transformation from the point of view of the observer to the new frame of the particle is the product of a boost with velocity $(v, \frac{\delta v}{\cosh(\mu)}, 0)$

$$L_{v,\delta} = \begin{pmatrix} \cosh(\mu) & -\frac{v}{c}\cosh(\mu) & -\frac{\delta v}{c} & 0 \\ -\frac{v}{c}\cosh(\mu) & \cosh(\mu) & (\cosh(\mu)-1)\frac{\delta v}{v\cosh(\mu)} & 0 \\ -\frac{\delta v}{c} & (\cosh(\mu)-1)\frac{\delta v}{v\cosh(\mu)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and a rotation about the z axis with small angle $\delta\theta$. Show that $L_{v,\delta}$ is a Lorentz transformation (to first order in δv), and find what the angle $\delta\theta$ is in terms of v and δv.

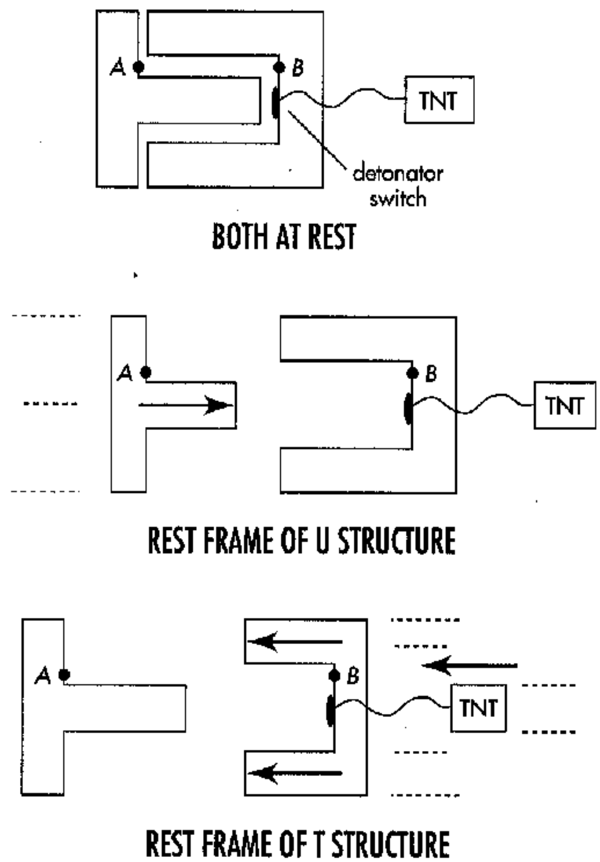
Note that the expression for a general rotation about the z axis is

$$R_{z,\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) & 0 \\ 0 & -sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ie show that

$$L_{\delta v} L_v = R_{z,\delta\theta} L_{v,\delta v}$$

for the appropriate $\delta\theta$ and keeping terms only to first order (linear order) in δv. This angle of rotation is the Thomas Precession angle.



EXERCISE 6-5. ***Both at rest:*** *The leg of the T almost reaches the detonator switch when both the T and the U are at rest. Points* A *and* B *are used in part b of the exercise.* ***Rest frame of U structure:*** *The leg of the moving T is Lorentz contracted in the rest frame of the U. Does this mean that the explosion will not take place?* ***Rest frame of T structure:*** *The legs of the moving U are Lorentz-contracted in the rest frame of the T. Does this mean explosion will take place?*

Note that if you calculating to first order in some quantity, all terms which are of order that quantity squared may be neglected.

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[6]– see the end

To show the first transformation, this is just the Lorentz transformation expressed in matrix form. Ie, they should show that

$$L_v \bar{x} \tag{14}$$

just gives the usual transformation

$$c\tilde{t} = cosh(\mu)ct - sinh(\mu)x \tag{15}$$
$$\tilde{x} = cosh(\mu)x - sinh(\mu)ct \tag{16}$$
$$\tilde{y} = y \tag{17}$$
$$\tilde{z} = z \tag{18}$$

The transformation in the y direction is exactly the same but with x replaced by y and vice versa. However the additional feature is that we assume that the velocity in the y direction is very small. Since $cosh(\mu_{\delta v}) = \frac{1}{\sqrt{1-\frac{\delta v^2}{c^2}}}$ for δv very small, and keeping only terms to linear order in δv, $cosh(\mu_{\delta v}) = 1$ and $sinh(\mu_{\delta v}) = \frac{\delta v}{c}$. Thus the matrix for the transformation in the y direction is just $L_{\delta v}$.

$$L_\delta = \begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\delta v}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

Then

$$L_\delta L_v \tag{20}$$
$$= \begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\delta v}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh(\mu) & -\frac{v}{c}\cosh(\mu) & 0 & 0 \\ -\frac{v}{c}\cosh(\mu) & \cosh(\mu) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$
$$= \begin{pmatrix} \cosh(\mu) & -\frac{v}{c}\cosh(\mu) & -\frac{\delta v}{c} & 0 \\ -\frac{v}{c}\cosh(\mu) & \cosh(\mu) & (\cosh(\mu)-1)\frac{\delta v}{v\cosh(\mu)} & 0 \\ -\frac{\delta v}{c} & (\cosh(\mu)-1)\frac{\delta v}{v\cosh(\mu)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

by simple matrix multiplication.

Now we can show that this is a Lorentz transformation in two ways. In the first we can show that

$$L^T G L = G \tag{23}$$

for this matrix directly, or we can recall that if both L_v and L_δ are Lorentz transforms, then the product also is.

$$\begin{aligned}(L_\delta L_v)^T G(L_\delta L_v) &= {L_v}^T {L_\delta}^T G(L_\delta L_v) = {L_v}^T({L_\delta}^T G L_\delta)L_v & (24)\\ &= L_v^T G L_v = G & (25)\end{aligned}$$

Let me do the latter first. Now we have shown in class that L_v is a Lorentz transformation. But is L_δ?

$$L_\delta^T G L_\delta = L_\delta G L_\delta \tag{26}$$

since L_{delta} is symmetric. We have

$$\begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0\\ 0 & 1 & 0 & 0\\ -\frac{\delta v}{c} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0\\ 0 & 1 & 0 & 0\\ -\frac{\delta v}{c} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} -1 & 0 & +\frac{\delta v}{c} & 0\\ 0 & 1 & 0 & 0\\ -\frac{\delta v}{c} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & -\frac{\delta v}{c} & 0\\ 0 & 1 & 0 & 0\\ -\frac{\delta v}{c} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{28}$$

$$= \begin{pmatrix} -1-\frac{\delta v^2}{c^2} & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1+\frac{\delta v^2}{c^2} & 0 & (29)\\ 0 & 0 & 0 & 1 & \end{pmatrix} \tag{30}$$

But we explicitly kept only terms to linear order in δv in the original expression. Thus, we cannot keep the terms of order $\frac{\delta v^2}{c^2}$ here, since we have already thrown away terms of exactly this form. Thus we set all terms of the form δv^2 to zero, and the final matrix then is just G, and L_δ is a Lorentz transformation.

Now, we need to show that we can find an angle theta such that the two expressions are equal. We could multiply the matrix by the rotation matrix but the strong suspicion is that the rotation angle θ will be very small. After all if $\delta v \to 0$, then the rotation angle will have to be zero as well. Thus our suspicion is that since we threw away all terms of order δv^2 we can also do the same for θ as well. If we throw away all terms of order θ^2 then $sin(\theta) = \theta$ and $cos(\theta) = 1$.

and the rotation matrix is

$$R \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

Multiplying $L_{\delta v} L_v$ by the rotation matrix we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} L_{v,\delta v} = \begin{pmatrix} \cosh(\mu) & -\frac{v}{c}\cosh(\mu) & -\frac{\delta v}{c} & 0 \\ -\frac{v}{c}\cosh(\mu) - \delta v\theta & \cosh(\mu) + \delta v\theta\frac{cosh(\mu)-1}{vcosh(\mu)} & \delta v\theta\frac{(\cosh(\mu)-1)}{v\cosh(\mu)} & 0 \\ \frac{v}{c}cosh(\mu)\theta - \frac{\delta v}{c} & \delta v\frac{(\cosh(\mu)-1)}{v\cosh(\mu)} & -\theta\delta v(\frac{cosh(\mu)-1}{vcosh(\mu)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

Now, since we suspect that θ is of order δv we must throw away all terms with $\theta\delta v$ since they will be of order δv^2. Equating this with $L_{\delta v} L_v$ we have from the 1st column third row

$$\frac{v}{c}cosh(\mu)\theta - \frac{\delta v}{c} = -\frac{deltav}{c}cosh(\mu) \tag{33}$$

or

$$\theta = -\delta v\frac{cosh(\mu) - 1}{vcosh(\mu)} \tag{34}$$

Similarly we get exactly the same solution from the 3rd colum, second row. From the second column third row we have

$$-\theta cosh(\mu) + \delta v\frac{cosh(\mu) - 1}{vcosh(\mu)} = \frac{\delta v}{c}\frac{v}{c}cosh(\mu) \tag{35}$$

or

$$\begin{aligned} \theta &= \frac{\delta v}{c}\left(c\frac{cosh(\mu) - 1}{vcosh(\mu)^2} - \frac{v}{c}\right) && (36) \\ &= \delta v\left(\frac{cosh(\mu) - 1 - cosh(\mu)^2 + cosh(\mu)^2(1 - \frac{v^2}{c^2}}{vcosh(\mu)^2}\right) && (37) \\ &= \delta v\left(\frac{cosh(\mu) - 1 - cosh(\mu)^2 + 1}{vcosh(\mu)^2}\right) && (38) \\ &= -\delta v\left(\frac{cosh(\mu) - 1}{vcosh(\mu)}\right) && (39) \end{aligned}$$

where we have remebered that $cosh(\mu) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

[Marking. Out of 6. 2 for showing these are Lorentz transforms. 1 for recongising that δv^2 is zero, One for getting the Matrix which is the product of rotation times $L_{v,\delta v}$, one for the linearization of the rotation matrix (cos and sin) and one for finding θ in terms of δv.]